Lecture at JAEA 1/23/2017 RIKEN

Introduction to Topological Insulators

Kentaro Nomura (IMR, Tohoku)

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Introduction to Topological Insulators

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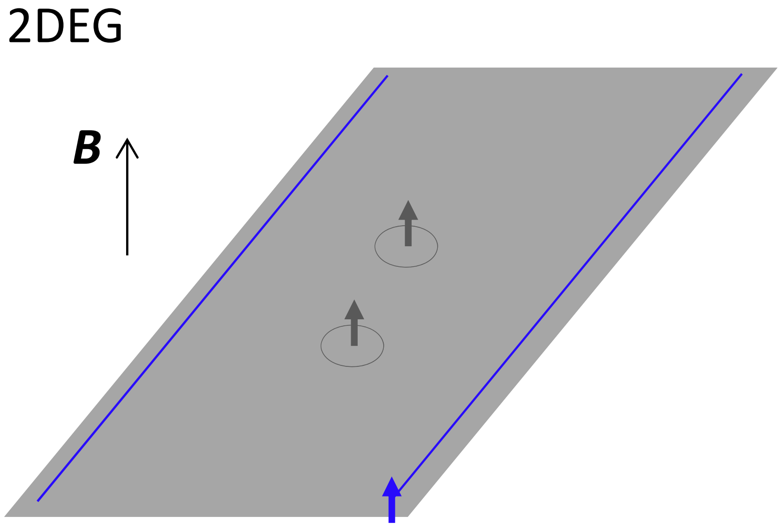
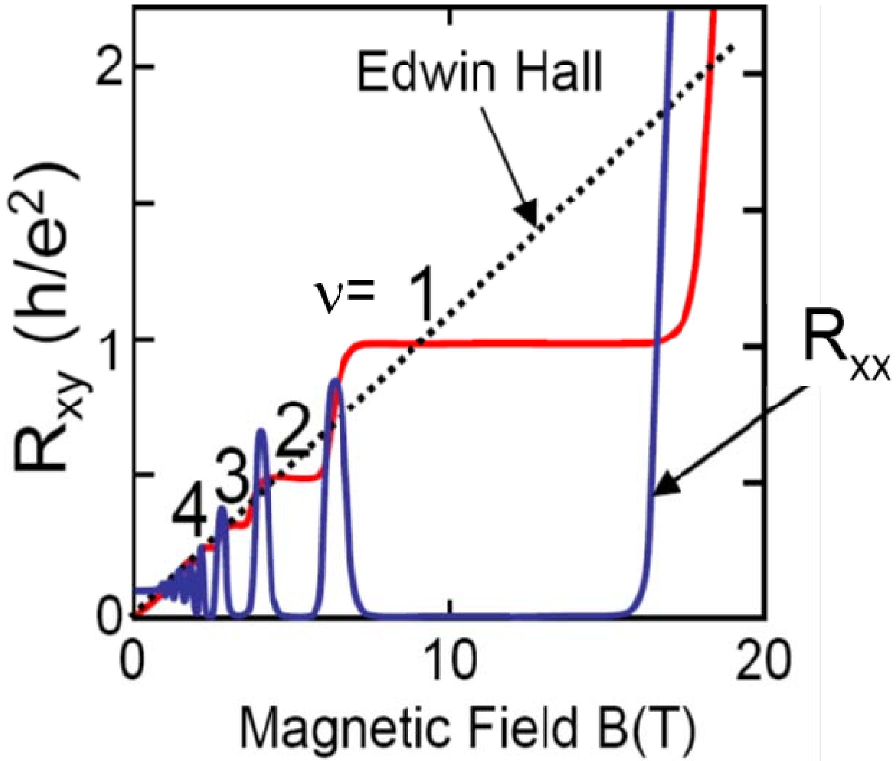
outline

- Quantum Hall effect
- Z2 topological insulators
- Electromagnetic responses

Quantum Hall effects



K. von Klitzing 1980

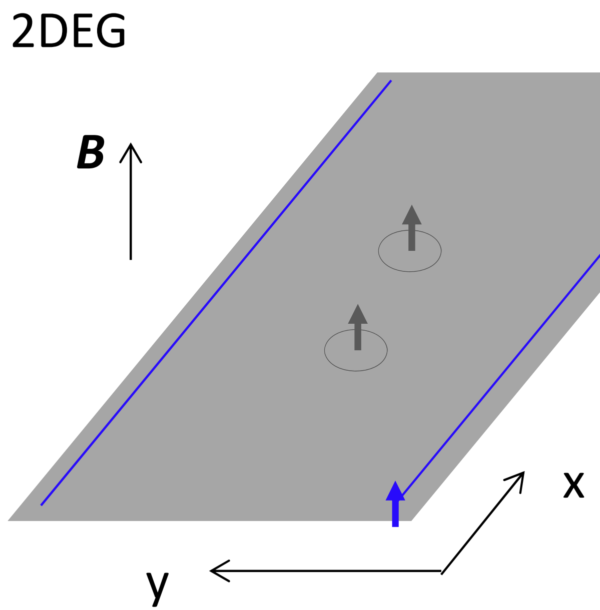


Hall conductivity

$$\sigma_{xy} = R_{xy}^{-1} = N \frac{e^2}{h}$$

$N \in \boldsymbol{Z} = \{..,-1, 0, 1, 2, ...\}$

Quantum Hall effects



$$H = \frac{1}{2m}\left(\boldsymbol{p} + e\boldsymbol{A}(\boldsymbol{r})\right)^2$$
$$= \frac{1}{2m}\left([-i\partial_x + e(-By)]^2 - \partial_y^2\right)$$

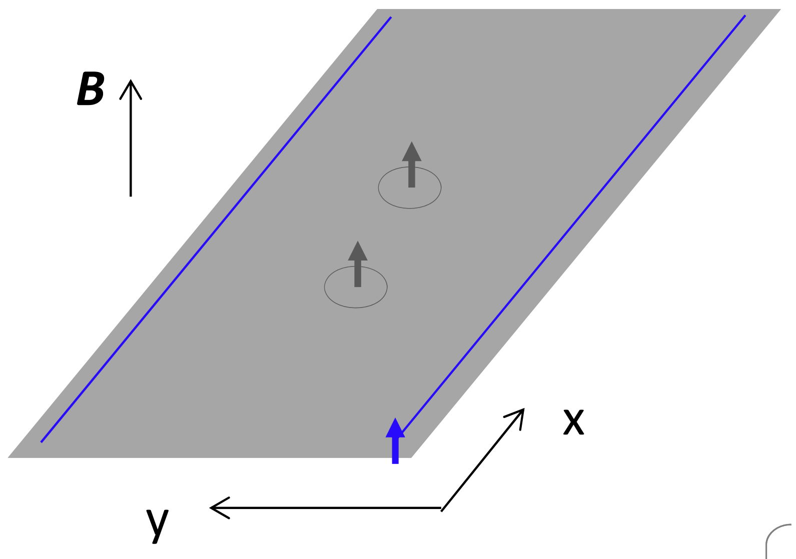
Landau gauge

$$\boldsymbol{A} = (-By, 0, 0)$$

translational symmetry in x-direction

Quantum Hall effects

2DEG



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$$= \frac{1}{2m}\left([-i\partial_x + e(-By)]^2 - \partial_y^2\right)$$

$$\begin{cases} -i\partial_x \to k_x = -\frac{2\pi}{L_x} n \\ y_n = -\ell_B^2 k_x = n\Delta y \\ \Delta y = \phi_0/(|B|L_x) \end{cases}$$

$\ell_B = \sqrt{\frac{\hbar c}{eB}}$

Landau gauge

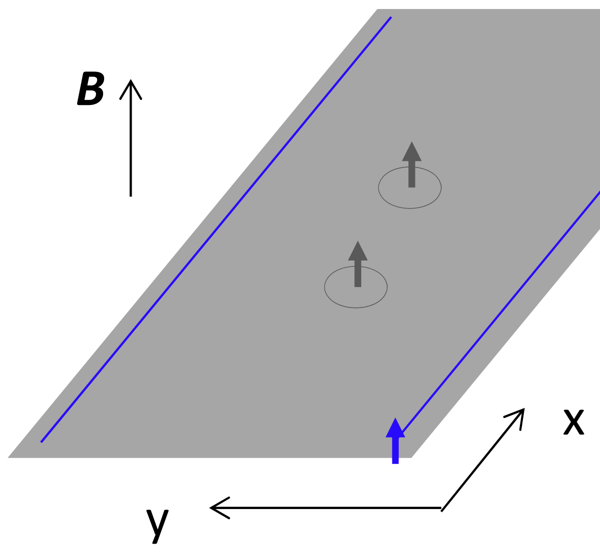
$$\boldsymbol{A} = (-By, 0, 0)$$

translational symmetry in x-direction

$$H = \frac{m\omega_c^2}{2}(y - y_n)^2 - \frac{1}{2m}\partial_y^2$$

n=1,2, ..., N_Φ

Quantum Hall effects



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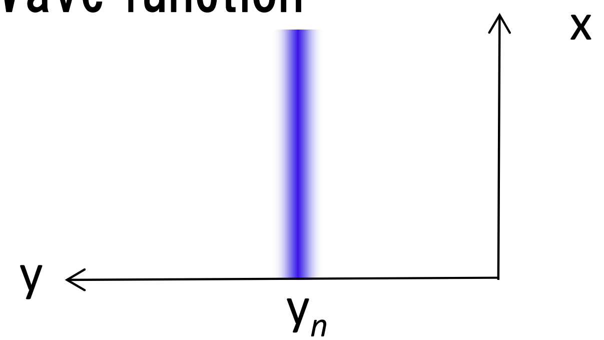
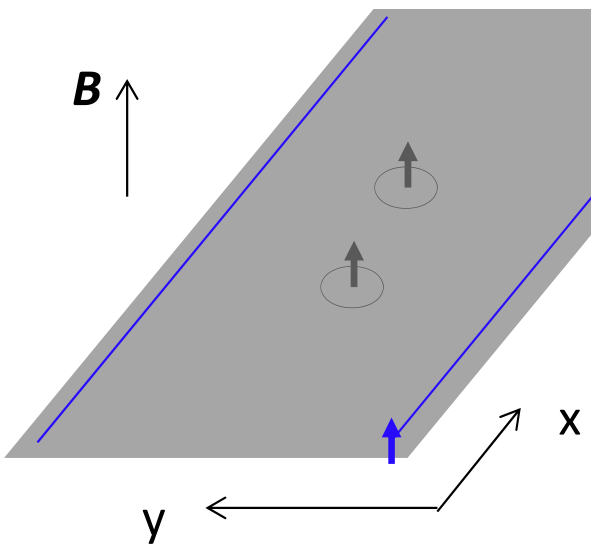


$n=1,2,\ldots,N_\Phi$

$$H = \frac{m\omega_c^2}{2}(y - y_n)^2 - \frac{1}{2m}\partial_y^2$$

Quantum Hall effects

2DEG



$$H = \frac{1}{2m}\left(\boldsymbol{p} + e\boldsymbol{A}(\boldsymbol{r})\right)^2$$

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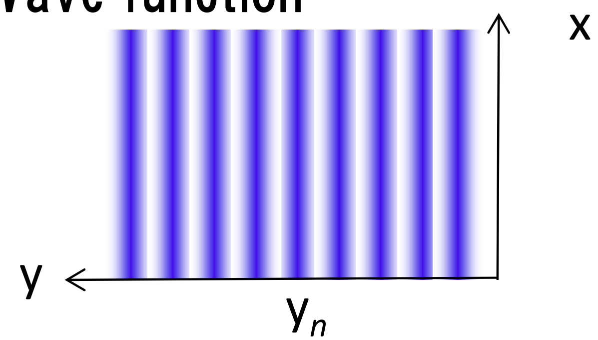
$$\ell_B = \sqrt{\frac{\hbar c}{eB}}$$

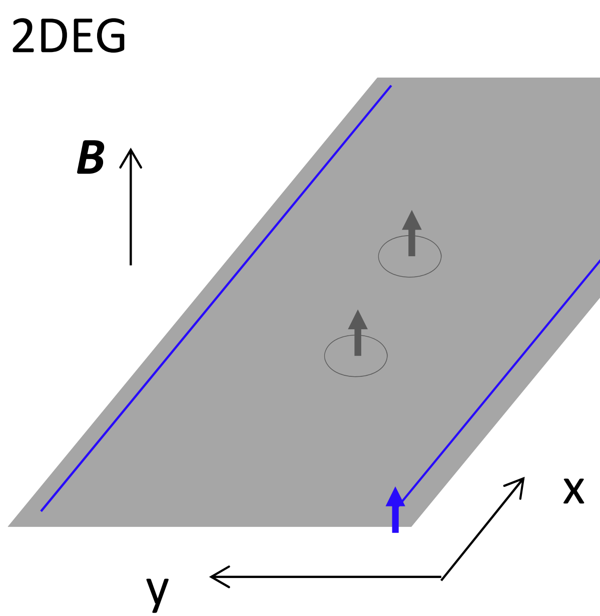
Wave function



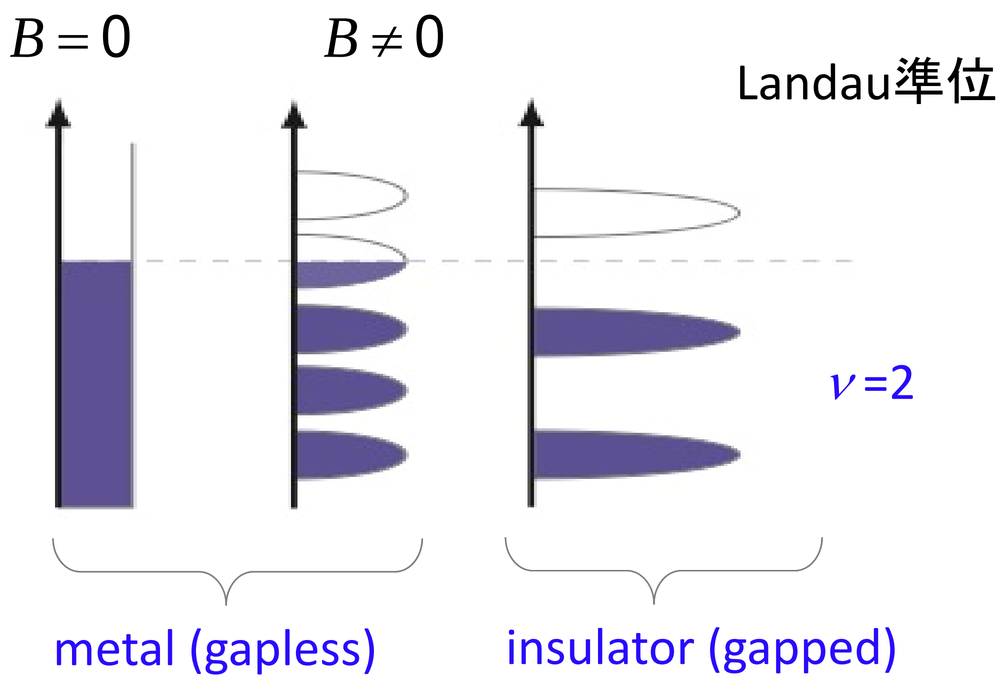
n=1,2, ..., N_Φ

$$H = \frac{m\omega_c^2}{2}(y - y_n)^2 - \frac{1}{2m}\partial_y^2$$

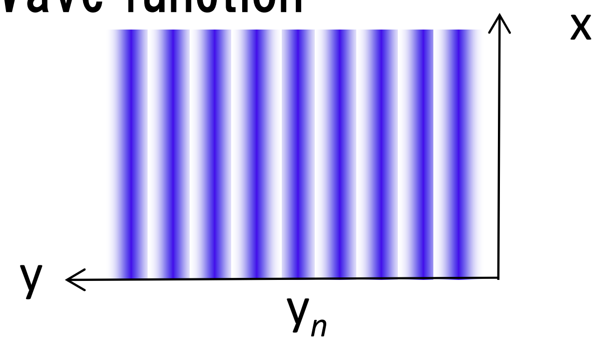
Quantum Hall effects



$$E_N = \hbar\omega_c(N + 1/2)$$



Wave function



n=1,2, ..., N_Φ

$$H = \frac{m\omega_c^2}{2}(y - y_n)^2 - \frac{1}{2m}\partial_y^2$$

Quantum Hall effects

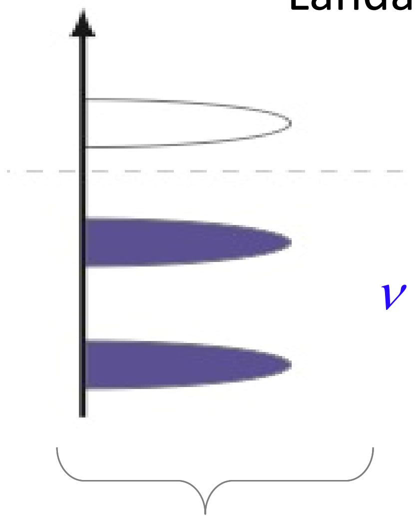
Insulators (gapped)

↓

Current does not flow

$$\boldsymbol{j} = 0 \ \ (?)$$

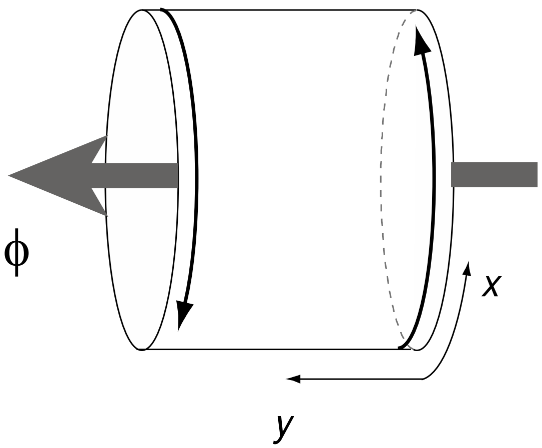
$$E_N = \hbar\omega_c (N + 1/2)$$



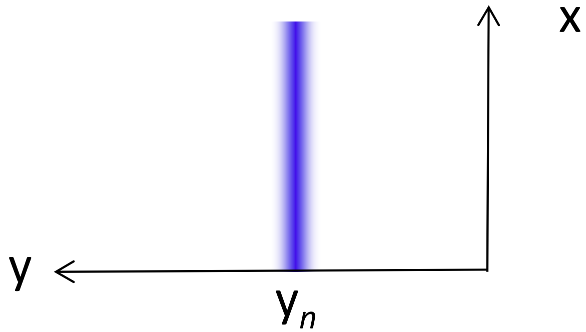
Quantized Hall current is carried in the ground state

Quantum Hall effects

Laughlin (1982)



$$H = \frac{1}{2m}\left(\boldsymbol{p} + e[\boldsymbol{A} + \boldsymbol{A}_\phi]\right)^2$$
$$= \frac{1}{2m}\left([-i\partial_x + e(-By + \tfrac{\phi}{L_x})]^2 - \partial_y^2\right)$$
$$= \frac{m\omega_c^2}{2}\left(y - y_n + \frac{\phi}{\phi_0}\Delta y\right)^2 - \frac{1}{2m}\partial_y^2$$

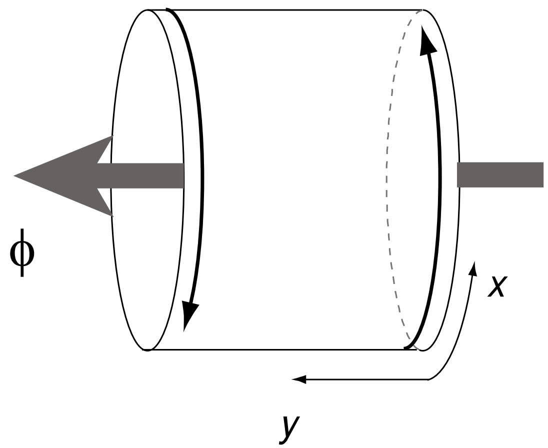


$$y_n = n\Delta y$$
$$\Delta y = L_y / N_\Phi$$

n=1,2, ..., N_Φ

Quantum Hall effects

Laughlin (1982)

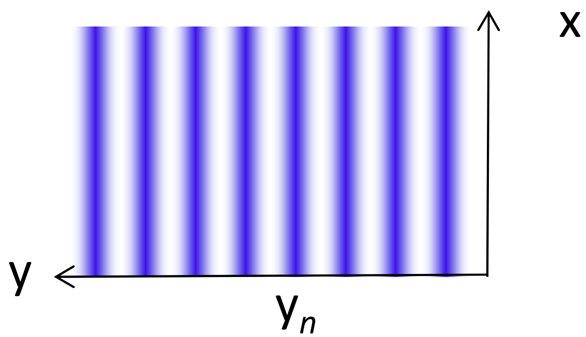


$$t = 0 \to T, \quad \phi = 0 \to \phi_0 \quad \Big\} \quad E_x L_x = \frac{\phi_0}{T}$$

Faraday's law

$$j_y = \frac{(-\nu e)}{L_x T} = \frac{-\nu e}{(h / eE_x)}$$

$$= -\nu \frac{e^2}{h} E_x$$



n=1,2, …, N_Φ

Hall conductivity

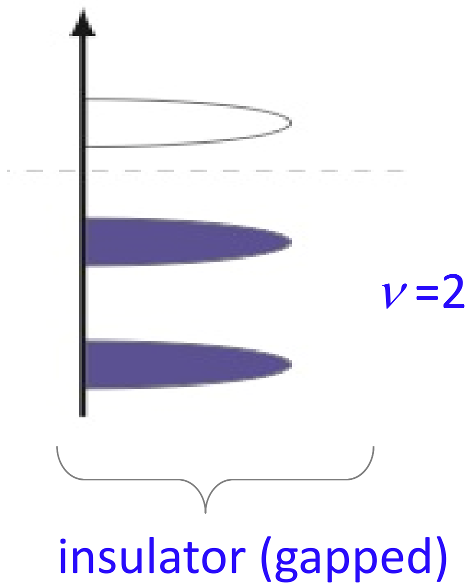
$$\sigma_{xy} = -\sigma_{yx} = \nu \frac{e^2}{h}$$

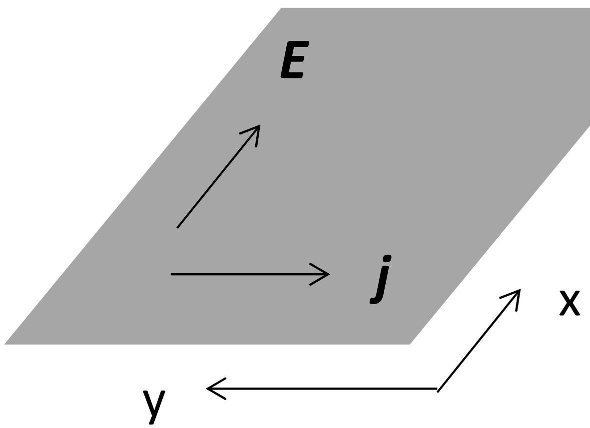
Quantum Hall effects

Quantum Hall insulators

↓

The ground state carries the current

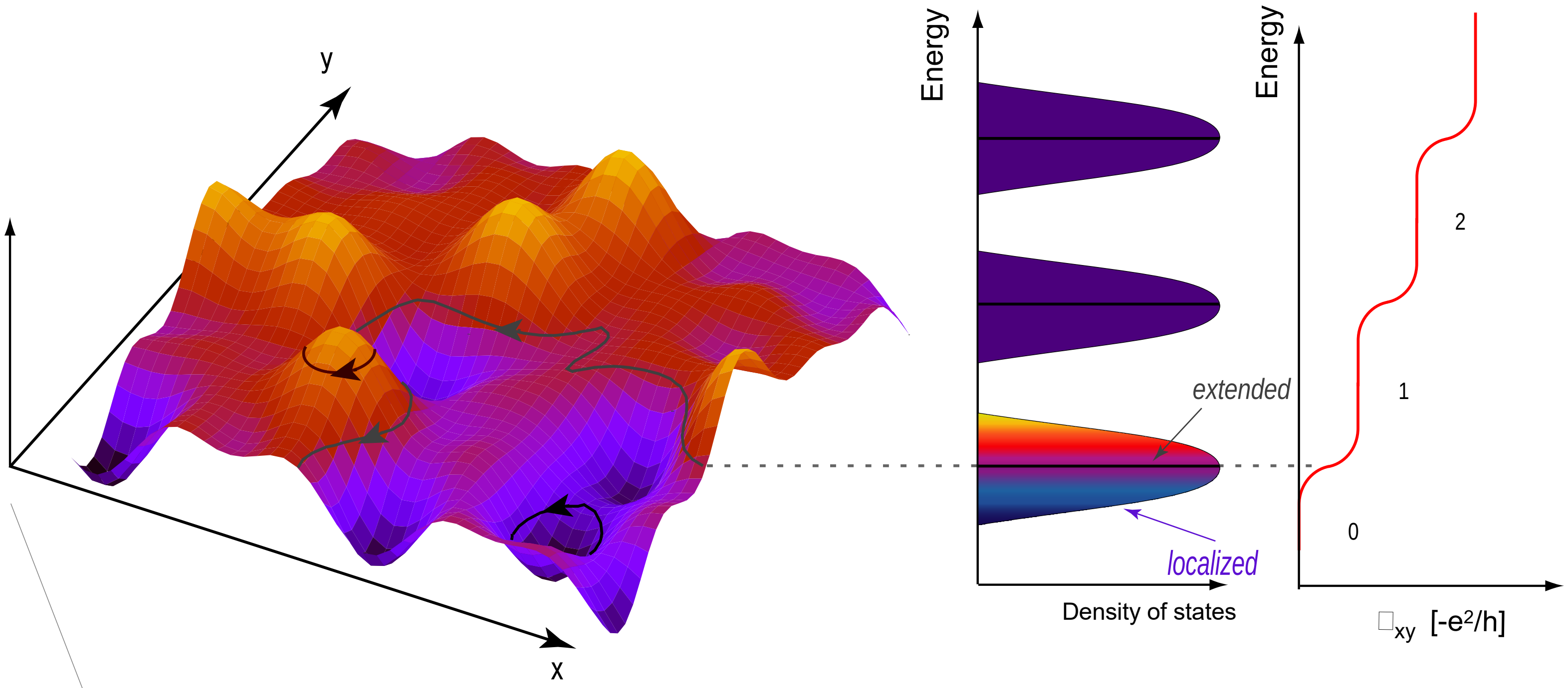


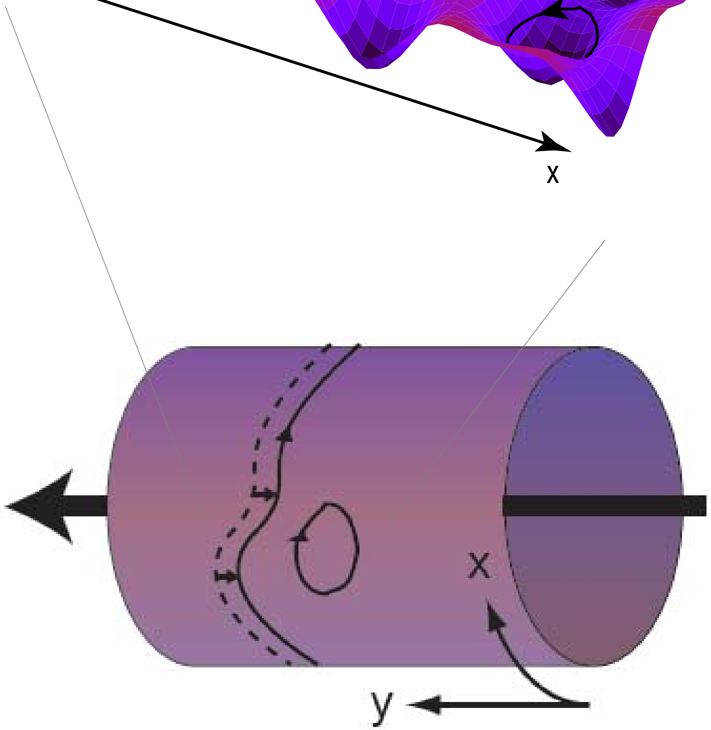


$$\sigma_{xy} = \nu \frac{e^2}{h}$$

No Joule heating

Disorder effects

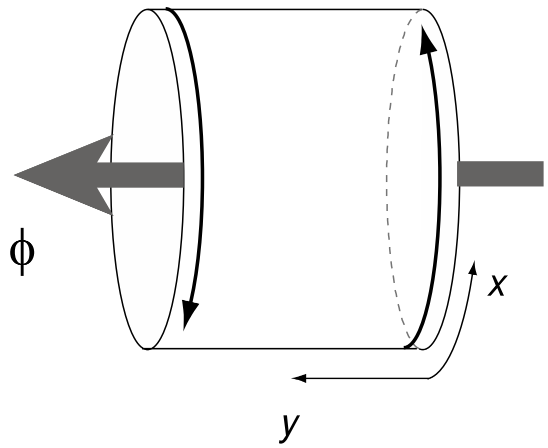


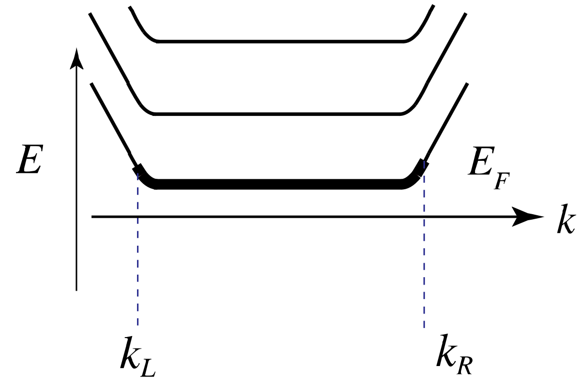


$$j_y = \frac{(-\nu e)}{L_x T} = \frac{-\nu e}{(h/eE_x)}$$

$$= -\nu \frac{e^2}{h} E_x$$

Edge states





potential term ↓

$$H = \frac{1}{2m}\left([-i\partial_x + e(-By)]^2 - \partial_y^2\right) + U(y)$$

$$\cong -\frac{1}{2m}\partial_y^2 + \frac{m\omega_c^2}{2}(y - y_n)^2 + U(y_n)$$

$$E_N(k_x) = \hbar\omega_c(N + 1/2) + U(y_n)$$

$$= \hbar\omega_c(N + 1/2) + U(-\ell_B^2 k_x)$$

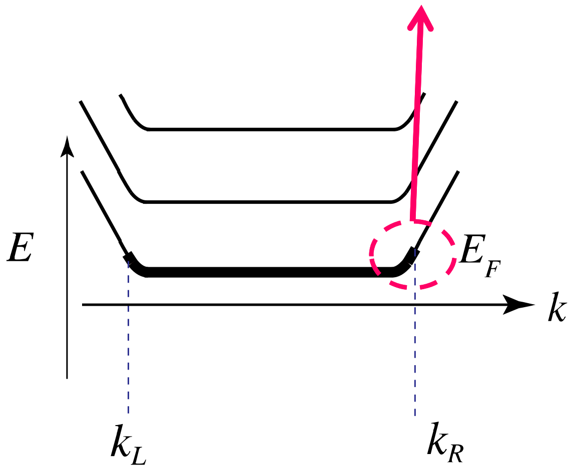
↑

$$y_n = -\ell_B^2 k_x$$

Edge states

$$E(k) \cong E_F + v_F (k - k_F^R) \qquad v_F \equiv \left.\frac{dE(k)}{dk}\right|_{k=k_F^R} > 0$$

-right moving modes



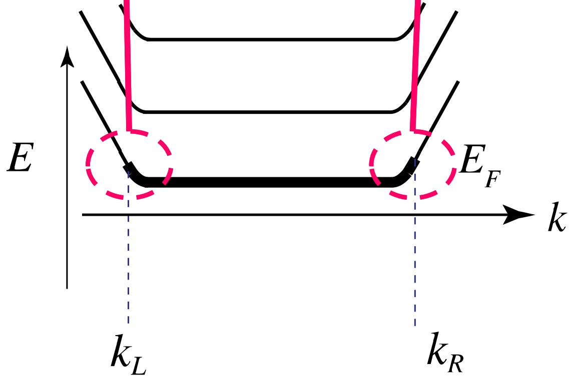
$$E_N(k_x) = \hbar\omega_c(N + 1/2) + U(y_n)$$
$$= \hbar\omega_c(N + 1/2) + U(-\ell_B^2 k_x)$$

$$y_n = -\ell_B^2 k_x$$

Edge states

$$E(k) \cong E_F - v_F(k - k_F^L)$$

$$E(k) \cong E_F + v_F(k - k_F^R)$$



$$-v_F \equiv \left.\frac{dE(k)}{dk}\right|_{k=k_F^L} < 0$$

-left moving modes

$$v_F \equiv \left.\frac{dE(k)}{dk}\right|_{k=k_F^R} > 0$$

-right moving modes

$$E_N(k_x) = \hbar\omega_c(N + 1/2) + U(y_n)$$
$$= \hbar\omega_c(N + 1/2) + U(-\ell_B^2 k_x)$$

$$y_n = -\ell_B^2 k_x$$

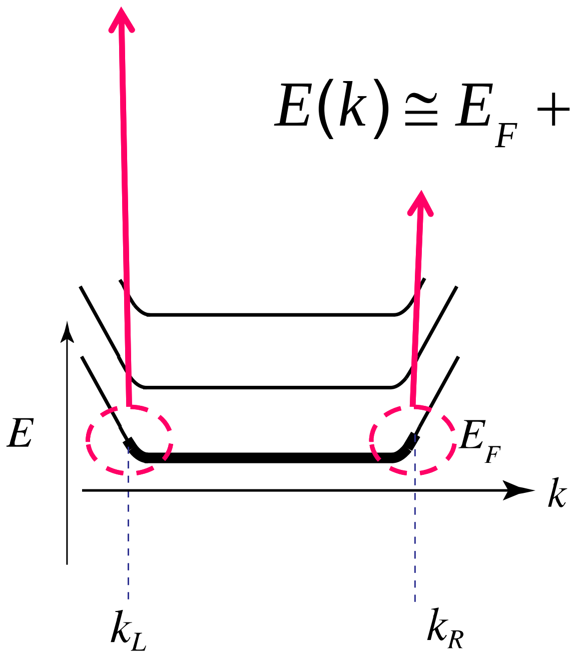
Edge states

$$E(k) \cong E_F - v_F (k - k_F^L)$$

$$E(k) \cong E_F + v_F (k - k_F^R)$$

$$\varphi_L(x,t) = e^{ik_F^R x} \psi_L(x,t)$$

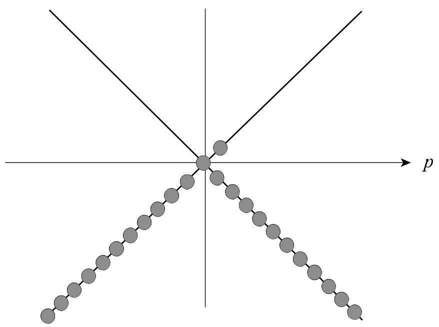
$$\varphi_R(x,t) = e^{ik_F^R x} \psi_R(x,t)$$



$$(i\partial_t + eA_0 + i\partial_x - eA_x)\psi_R = 0$$

$$(i\partial_t + eA_0 - i\partial_x + eA_x)\psi_L = 0$$

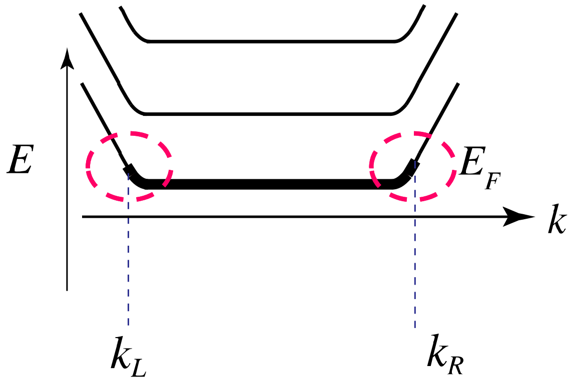
Edge states



$$N_R = \int dx \ \psi_R^+ \psi_R$$

$$N_L = \int dx \ \psi_L^+ \psi_L$$

2nd quantization formalism



$$\frac{d}{dt}\left(\frac{N_R - N_L}{2}\right) = \nu \frac{-e}{h} \int dx \ E_x$$

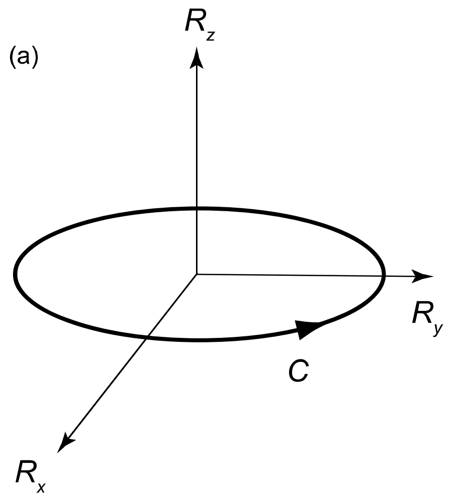
anomaly equation

$$(i\partial_t + eA_0 + i\partial_x - eA_x)\psi_R = 0$$

$$(i\partial_t + eA_0 - i\partial_x + eA_x)\psi_L = 0$$

Berry's phase

$$\gamma_n[C] = \int_0^T dt \ \dot{\boldsymbol{R}}(t) \cdot i\langle n, \boldsymbol{R}(t)|\boldsymbol{\nabla}_R|n, \boldsymbol{R}(t)\rangle$$
$$= \oint_C d\boldsymbol{R} \cdot i\langle n, \boldsymbol{R}|\boldsymbol{\nabla}_R|n, \boldsymbol{R}\rangle$$
$$\equiv -\oint_C d\boldsymbol{R} \cdot \boldsymbol{A}_n(\boldsymbol{R}) = -\int_S d\boldsymbol{S} \cdot \boldsymbol{B}_n(\boldsymbol{R})$$



$$|n,\boldsymbol{R}\rangle \rightarrow e^{i\gamma_n[C]}|n,\boldsymbol{R}\rangle$$

Berry’s phase

$$
\begin{aligned}
\gamma_n[C] &= \int_0^T dt\ \ \dot{\boldsymbol{R}}(t)\cdot i\langle n,\boldsymbol{R}(t)|\boldsymbol{\nabla}_{R}|n,\boldsymbol{R}(t)\rangle \\
&= \oint_C d\boldsymbol{R}\cdot i\langle n,\boldsymbol{R}|\boldsymbol{\nabla}_{R}|n,\boldsymbol{R}\rangle \\
&\equiv -\oint_C d\boldsymbol{R}\cdot \boldsymbol{A}_n(\boldsymbol{R}) \ = \ -\int_S d\boldsymbol{S}\cdot \boldsymbol{B}_n(\boldsymbol{R})
\end{aligned}
$$

$$\boldsymbol{A}_n(\boldsymbol{R}) = -i\langle n,\boldsymbol{R}|\boldsymbol{\nabla}_{R}|n,\boldsymbol{R}\rangle$$

Berry connection

$$\boldsymbol{B}_n(\boldsymbol{R}) = \boldsymbol{\nabla}_{R}\times\boldsymbol{A}_n(\boldsymbol{R})$$

Berry curvature

Berry's phase

$$\gamma_n[C] =$$

$$=$$

$$|n, \boldsymbol{R}\rangle' = e^{i\Lambda(\boldsymbol{R})}|n, \boldsymbol{R}\rangle$$

$$\begin{aligned} \boldsymbol{A}'_n(\boldsymbol{R}) &= -i\Big(\langle n, \boldsymbol{R}|e^{-i\Lambda(\boldsymbol{R})}\Big)\boldsymbol{\nabla}_{R}\Big(e^{i\Lambda(\boldsymbol{R})}|n, \boldsymbol{R}\rangle\Big) \\ &= \boldsymbol{A}_n(\boldsymbol{R}) + \boldsymbol{\nabla}_{R}\Lambda(\boldsymbol{R}) \end{aligned}$$

Gauge transformation

$$\equiv -\oint_C d\boldsymbol{R}\cdot\boldsymbol{A}_n(\boldsymbol{R}) = -\int_S d\boldsymbol{S}\cdot\boldsymbol{B}_n(\boldsymbol{R})$$

$$\boldsymbol{A}_n(\boldsymbol{R}) = -i\langle n, \boldsymbol{R}|\boldsymbol{\nabla}_{R}|n, \boldsymbol{R}\rangle$$

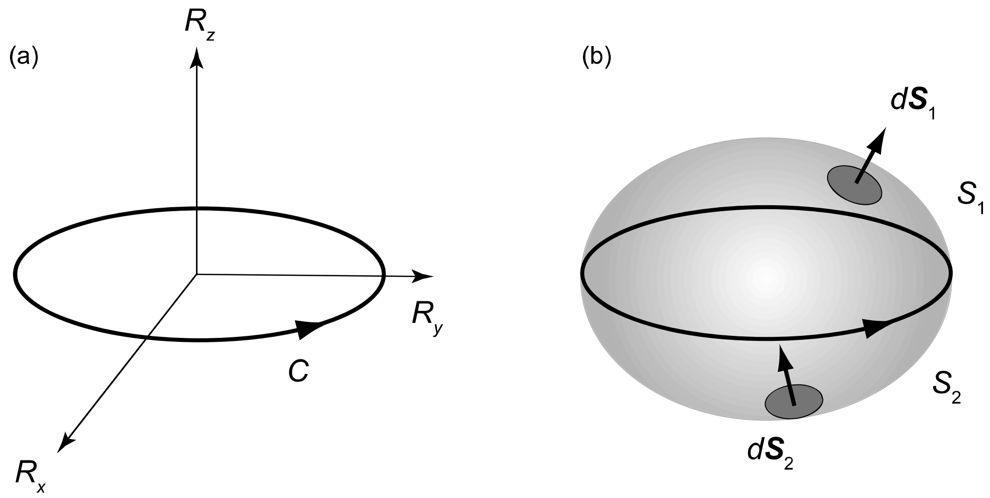
Berry connection

$$\boldsymbol{B}_n(\boldsymbol{R}) = \boldsymbol{\nabla}_{R} \times \boldsymbol{A}_n(\boldsymbol{R})$$

Berry curvature

Berry's phase

$$\begin{aligned}\gamma_n[C] &= \int_0^T dt \;\; \dot{\boldsymbol{R}}(t)\cdot i\langle n,\boldsymbol{R}(t)|\boldsymbol{\nabla}_R|n,\boldsymbol{R}(t)\rangle \\ &= \oint_C d\boldsymbol{R}\cdot i\langle n,\boldsymbol{R}|\boldsymbol{\nabla}_R|n,\boldsymbol{R}\rangle \\ &\equiv -\oint_C d\boldsymbol{R}\cdot\boldsymbol{A}_n(\boldsymbol{R}) \;=\; -\int_S d\boldsymbol{S}\cdot\boldsymbol{B}_n(\boldsymbol{R})\end{aligned}$$

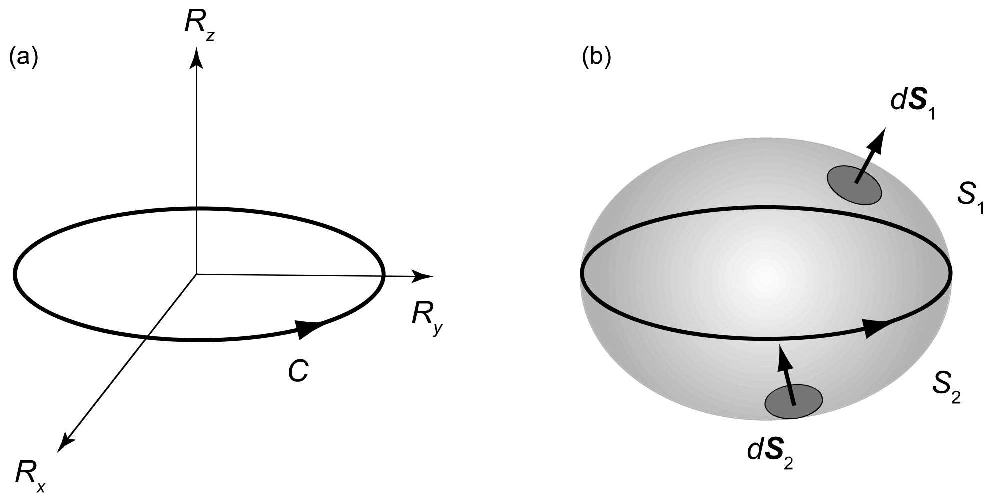


$$\gamma_n[C] \;=\; -\int_S d\boldsymbol{S}\cdot\boldsymbol{B}_n(\boldsymbol{R})$$

Berry's phase

$$\int_{S_1} d\boldsymbol{S}_1 \cdot \boldsymbol{B}_n(\boldsymbol{R}) = \int_{S_2} d\boldsymbol{S}_2 \cdot \boldsymbol{B}_n(\boldsymbol{R}) + 2\pi N, \qquad N \in \mathbb{Z}$$

$$\int_{S_1 - S_2} d\boldsymbol{S} \cdot \boldsymbol{B}_n(\boldsymbol{R}) = 2\pi N$$

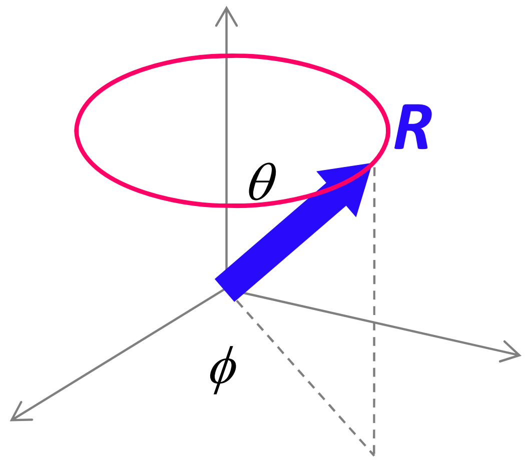


$$\gamma_n[C] \quad = \quad -\int_S d\boldsymbol{S} \cdot \boldsymbol{B}_n(\boldsymbol{R})$$

Berry's phase

$$H[\boldsymbol{R}] = \boldsymbol{R}\cdot\boldsymbol{\sigma} = \begin{pmatrix} R_z & R_x - iR_y \\ R_x + iR_y & -R_z \end{pmatrix}$$

$$|+,\boldsymbol{R}\rangle = e^{i\psi/2}\begin{pmatrix} e^{i\phi/2}\cos\frac{\theta}{2} \\ e^{-i\phi/2}\sin\frac{\theta}{2} \end{pmatrix}$$



$$\begin{aligned}
\boldsymbol{A}_+(\boldsymbol{R}) &= -i\langle +,\boldsymbol{R}|\boldsymbol{\nabla}_R|+,\boldsymbol{R}\rangle \\
&= -ie^{+i\frac{\psi}{2}}\Big(e^{+i\frac{\phi}{2}}\cos\frac{\theta}{2}, e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\Big) \\
&\quad \cdot e^{-i\frac{\psi}{2}}\begin{pmatrix} -\frac{i}{2}\boldsymbol{\nabla}(\psi+\phi)\,e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} - \frac{1}{2}\boldsymbol{\nabla}\theta\,e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ -\frac{i}{2}\boldsymbol{\nabla}(\psi-\phi)\,e^{+i\frac{\phi}{2}}\sin\frac{\theta}{2} + \frac{1}{2}\boldsymbol{\nabla}\theta\,e^{+i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \\
&= -i\Big[-\frac{i}{2}\boldsymbol{\nabla}\psi - \frac{i}{2}\boldsymbol{\nabla}\phi\Big(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\Big)\Big] \\
&= \frac{1}{2}\Big(-\boldsymbol{\nabla}\psi - \boldsymbol{\nabla}\phi\cos\theta\Big)
\end{aligned}$$

Berry's phase

$$\underline{\psi = -\phi}, \quad \boldsymbol{A}_{+}^{N}(\boldsymbol{R}) = \frac{1}{2}(+1-\cos\theta)\boldsymbol{\nabla}\phi = \frac{+1-\cos\theta}{2R\sin\theta}\mathbf{e}_{\phi}$$

$$\underline{\psi = +\phi}, \quad \boldsymbol{A}_{+}^{S}(\boldsymbol{R}) = \frac{1}{2}(-1-\cos\theta)\boldsymbol{\nabla}\phi = \frac{-1-\cos\theta}{2R\sin\theta}\mathbf{e}_{\phi}$$

$$\begin{aligned}
\boldsymbol{A}_{+}(\boldsymbol{R}) &= -i\langle +,\boldsymbol{R}|\boldsymbol{\nabla}_{R}|+,\boldsymbol{R}\rangle \\
&= -ie^{+i\frac{\psi}{2}}\left(e^{+i\frac{\phi}{2}}\cos\frac{\theta}{2}, e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\right) \\
&\qquad \cdot e^{-i\frac{\psi}{2}}\begin{pmatrix} -\frac{i}{2}\boldsymbol{\nabla}(\psi+\phi)\,e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} - \frac{1}{2}\boldsymbol{\nabla}\theta\, e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ -\frac{i}{2}\boldsymbol{\nabla}(\psi-\phi)\,e^{+i\frac{\phi}{2}}\sin\frac{\theta}{2} + \frac{1}{2}\boldsymbol{\nabla}\theta\, e^{+i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \\
&= -i\Big[-\frac{i}{2}\boldsymbol{\nabla}\psi - \frac{i}{2}\boldsymbol{\nabla}\phi\Big(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\Big)\Big] \\
&= \frac{1}{2}\Big(-\boldsymbol{\nabla}\psi - \boldsymbol{\nabla}\phi\cos\theta\Big)
\end{aligned}$$

Berry's phase

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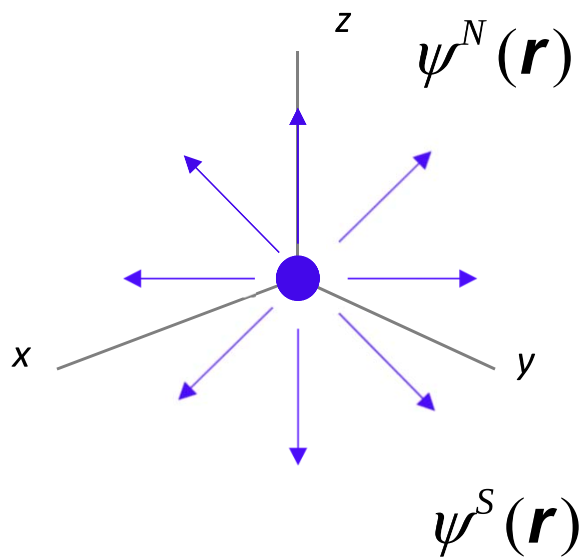
$$\underline{\psi = +\phi}, \quad \boldsymbol{A}_{+}^{S}(\boldsymbol{R}) = \frac{1}{2}(-1 - \cos\theta)\boldsymbol{\nabla}\phi = \frac{-1 - \cos\theta}{2R\sin\theta}\mathbf{e}_{\phi}$$

$$\begin{aligned} \boldsymbol{B}_{\pm}(\boldsymbol{R}) &= \boldsymbol{\nabla} \times \boldsymbol{A}_{\pm}^{N}(\boldsymbol{R}) = \boldsymbol{\nabla} \times \boldsymbol{A}_{\pm}^{S}(\boldsymbol{R}) \\ &= \pm\frac{1}{2}\ \frac{\boldsymbol{R}}{R^3} \end{aligned}$$



Berry's phase

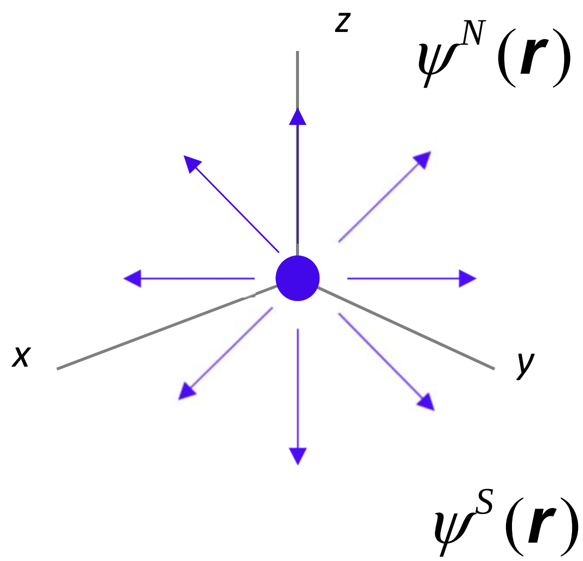
$$\underline{\psi = -\phi}, \quad \boldsymbol{A}_{+}^{N}(\boldsymbol{R}) = \frac{1}{2}(+1-\cos\theta)\boldsymbol{\nabla}\phi = \frac{+1-\cos\theta}{2R\sin\theta}\mathbf{e}_{\phi}$$

$$\underline{\psi = +\phi}, \quad \boldsymbol{A}_{+}^{S}(\boldsymbol{R}) = \frac{1}{2}(-1-\cos\theta)\boldsymbol{\nabla}\phi = \frac{-1-\cos\theta}{2R\sin\theta}\mathbf{e}_{\phi}$$

$$\begin{aligned} \boldsymbol{B}_{\pm}(\boldsymbol{R}) &= \boldsymbol{\nabla}\times\boldsymbol{A}_{\pm}^{N}(\boldsymbol{R}) = \boldsymbol{\nabla}\times\boldsymbol{A}_{\pm}^{S}(\boldsymbol{R}) \\ &= \pm\frac{1}{2}\ \frac{\boldsymbol{R}}{R^3} \end{aligned}$$

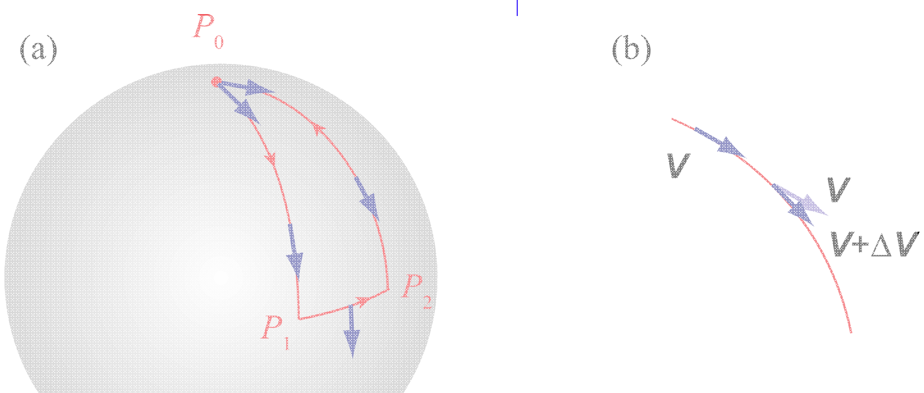
$$\int_{S_1-S_2} d\boldsymbol{S}\cdot\boldsymbol{B}_n(\boldsymbol{R}) = 2\pi N$$

$N = \pm 1$

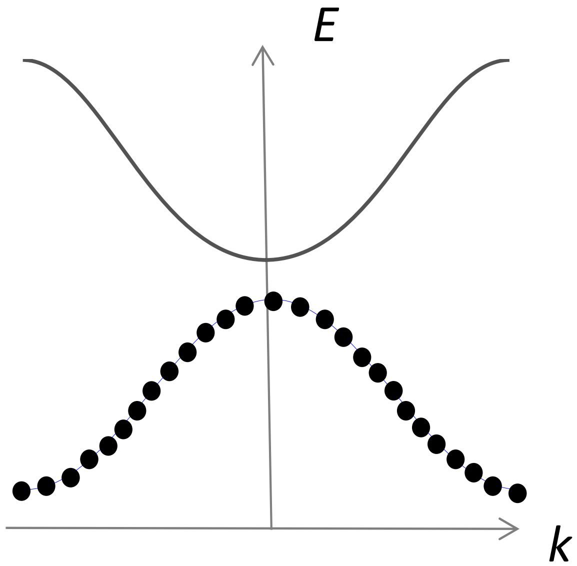


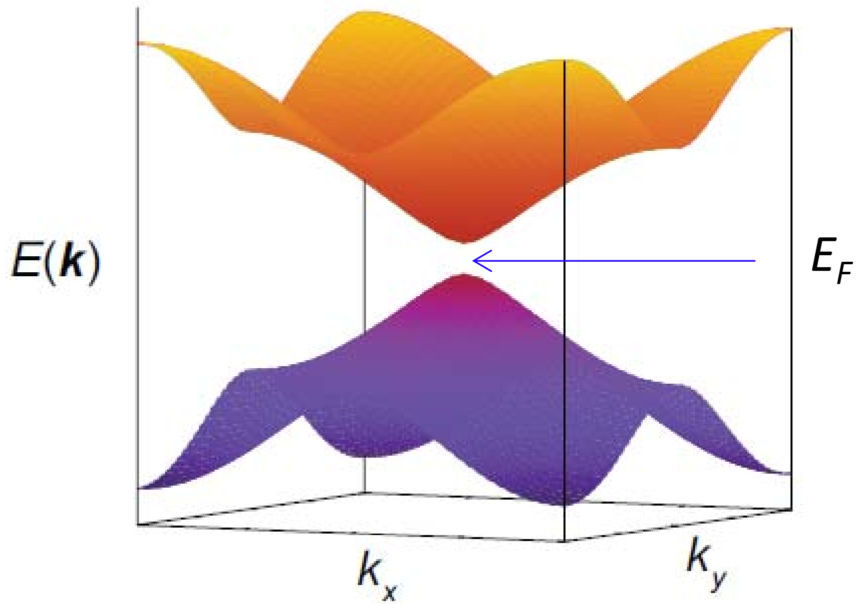
Geometry and Quantum Mechanics

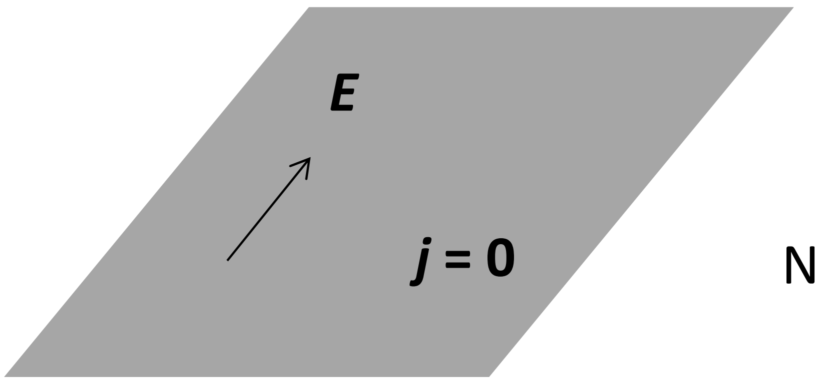
	Curved space	Quantum system
basis	$\hat{\mathbf{e}}_i(\boldsymbol{x})$	$\vert n,\boldsymbol{R}\rangle$
differential	$\partial_i V^j \to \partial_i V^j + \Gamma^j_{ik} V^k = D_i V^j$	$\partial_{\boldsymbol{R}} \to \partial_{\boldsymbol{R}} + i\boldsymbol{A} = D_{\boldsymbol{R}}$
connection	$\Gamma^k_{ji}(\boldsymbol{x}) = \hat{\mathbf{e}}^k(\boldsymbol{x})\partial_j \hat{\mathbf{e}}_i(\boldsymbol{x})$	$\boldsymbol{A}(\boldsymbol{R}) = -i\langle n,\boldsymbol{R}\vert\partial_{\boldsymbol{R}}\vert n,\boldsymbol{R}\rangle$
cuarvature	$(D_a D_b - D_b D_a)V^j = R^j_{iab} V^i$	$D_x D_y - D_y D_x = iB_z$



Trivial band insulator

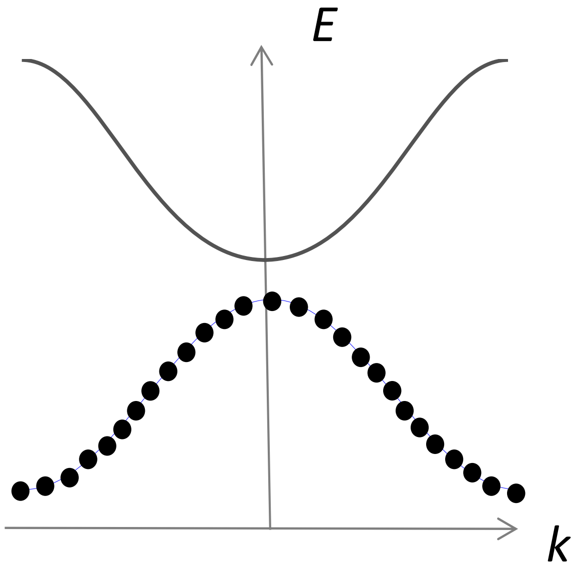


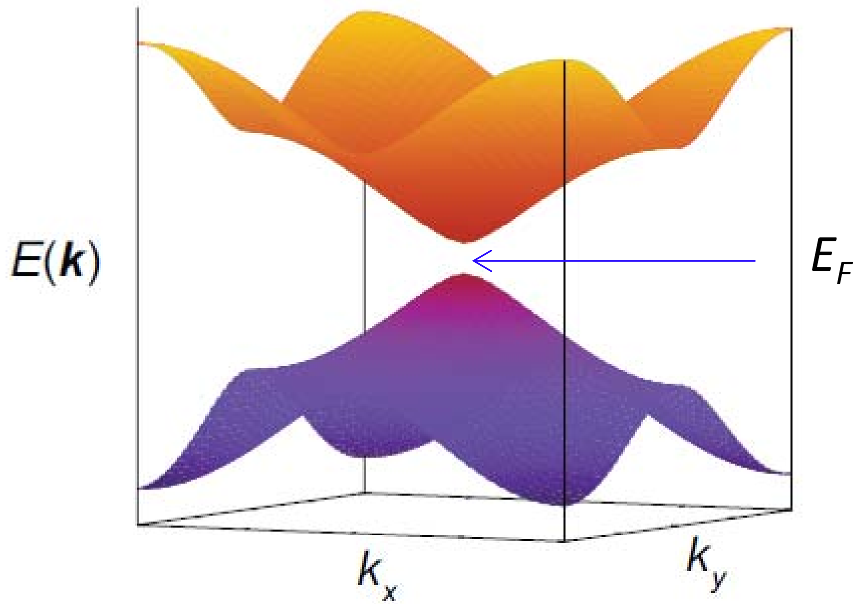


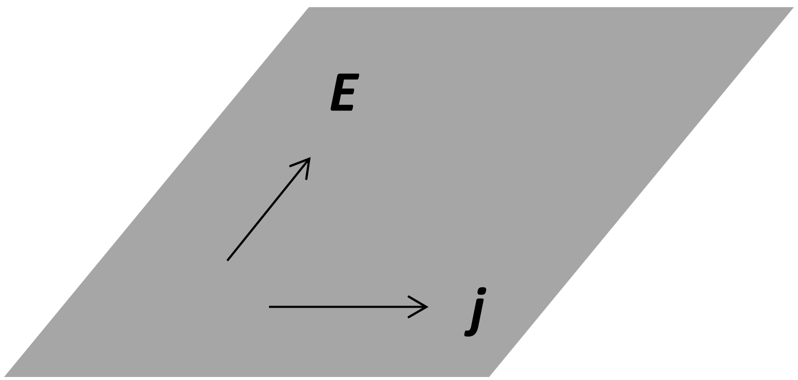


No current flows

Non-trivial band insulator



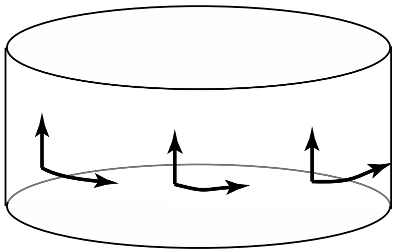




$$\sigma_{xy} = \frac{j_x}{E_y} = N\ \frac{e^2}{h}$$

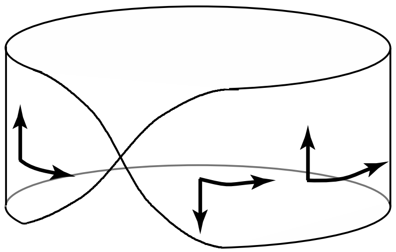
QHE

Topology?



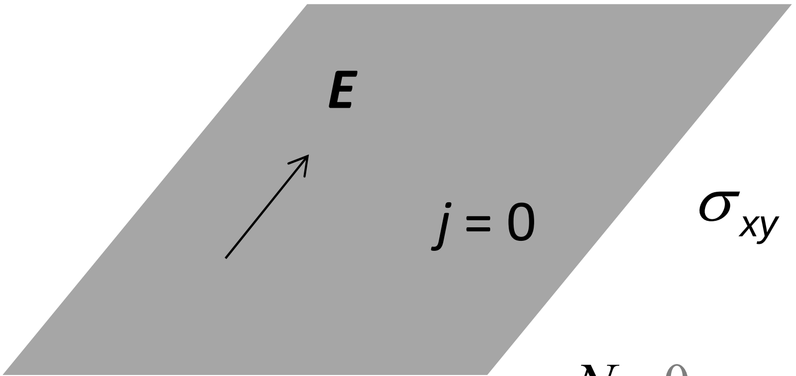
$N = 0$

topologically trivial



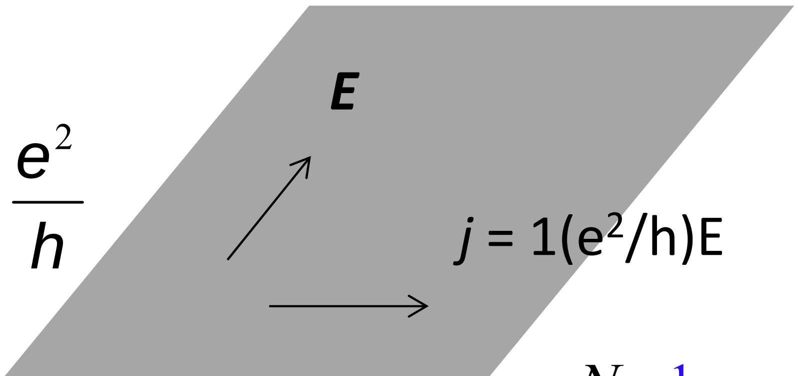
$N = 1$

topologically nontrivial



$N = 0$

$$\sigma_{xy} = \frac{j_x}{E_y} = N \frac{e^2}{h}$$



$N = 1$

Hall conductivity

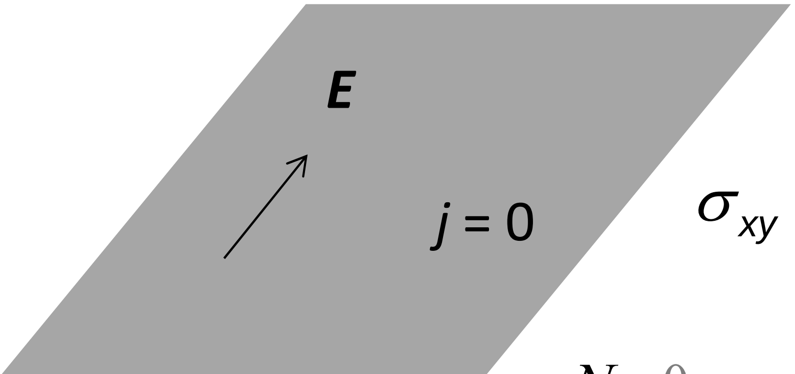
$$\sigma_{xy} = j_x / E_y$$

$N = 0$

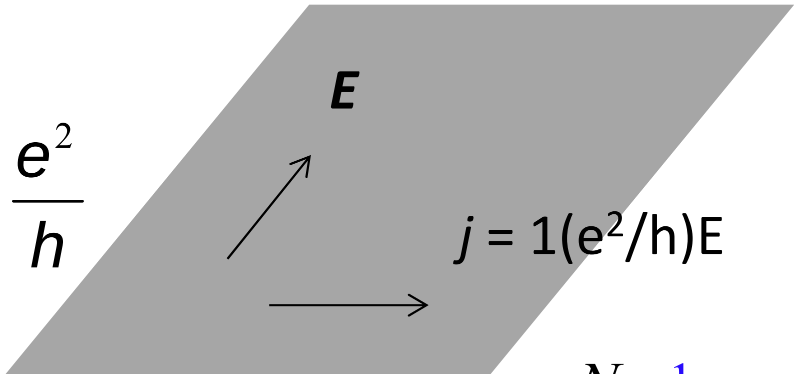
$N = 1$

topologically trivial

topologically nontrivial



$$\sigma_{xy} = \frac{j_x}{E_y} = N \frac{e^2}{h}$$



$N = 0$

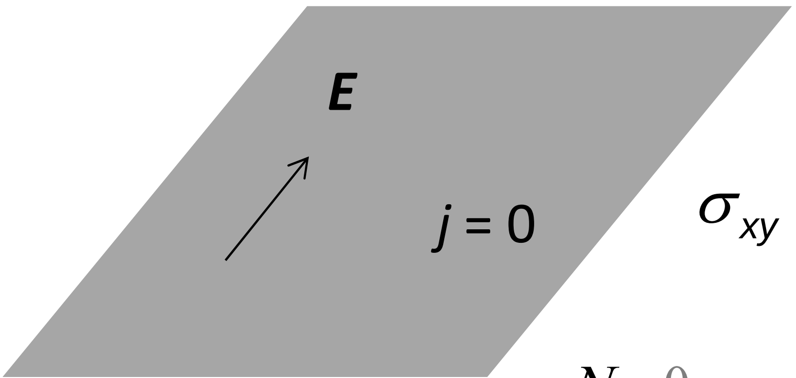
$N = 1$

Hall conductivity

$$\sigma_{xy} = j_x / E_y$$

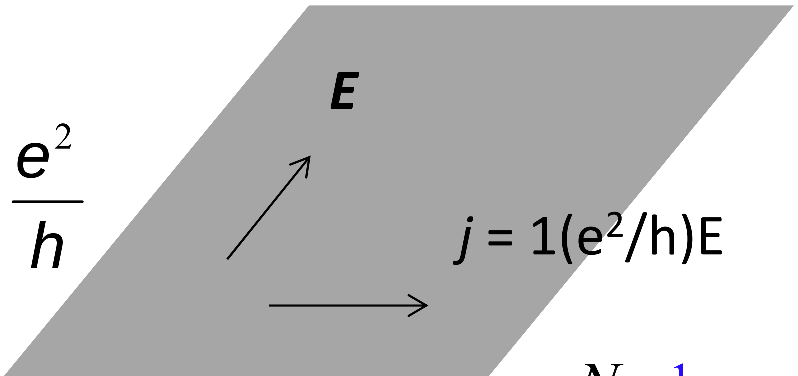
Perturbation theory: $H_E = H_0 - eE_y y$

$$|n\rangle_E = |n\rangle + \sum_{m(\neq n)} \frac{\langle m|(-eEy)|n\rangle}{E_n - E_m} |m\rangle + \cdots$$



$N = 0$

$$\sigma_{xy} = \frac{j_x}{E_y} = N \frac{e^2}{h}$$



$N = 1$

Hall conductivity

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Perturbation theory : $H_E = H_0 - eE_y y$

$$|n\rangle_E = |n\rangle + \sum_{m(\neq n)} \frac{\langle m|(-eEy)|n\rangle}{E_n - E_m}|m\rangle + \cdots$$

$$\begin{aligned}
\langle j_x \rangle_E &= \sum_n f(E_n) \langle n|_E \frac{ev_x}{L^2}|n\rangle_E \\
&= \langle j_x \rangle_{E=0} + \frac{1}{L^2}\sum_n f(E_n) \sum_{m(\neq n)} \\
&\qquad \left(\frac{\langle n|(ev_x)|m\rangle\langle m|(-eEy)|n\rangle}{E_n - E_m} + \frac{\langle n|(-eEy)|m\rangle\langle m|(ev_x)|n\rangle}{E_n - E_m} \right)
\end{aligned}$$

Hall conductivity

$$\sigma_{xy} = j_x / E_y$$

$$v_y = \frac{1}{i\hbar}[y, H]$$ Heisenberg equation

$$\langle m|v_y|n\rangle = \frac{1}{i\hbar}(E_n - E_m)\langle m|y|n\rangle$$

$$\begin{aligned}
\langle j_x\rangle_E &= \sum_n f(E_n)\langle n|_E \frac{ev_x}{L^2}|n\rangle_E \\
&= \langle j_x\rangle_{E=0} + \frac{1}{L^2}\sum_n f(E_n)\sum_{m(\neq n)} \\
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\end{aligned}$$

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&= \langle j_x\rangle_{E=0} + \frac{1}{L^2}\sum_n f(E_n)\sum_{m(\neq n)} \\
&\quad \left(\frac{\langle n|(ev_x)|m\rangle\langle m|(-eEy)|n\rangle}{E_n - E_m} + \frac{\langle n|(-eEy)|m\rangle\langle m|(ev_x)|n\rangle}{E_n - E_m} \right)
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$$\begin{aligned}\sigma_{xy} &= \frac{\langle j_x\rangle_E}{E} \\ &= -\frac{i\hbar e^2}{L^2}\sum_{n\neq m} f(E_n)\frac{\langle n|v_x|m\rangle\langle m|v_y|n\rangle - \langle n|v_y|m\rangle\langle m|v_x|n\rangle}{(E_n - E_m)^2}\end{aligned}$$

Hall conductivity

$$H|u_{n\mathbf{k}}\rangle = E_{n\mathbf{k}}|u_{n\mathbf{k}}\rangle$$

$$v_\mu = \frac{\partial\mathcal{H}(\mathbf{k})}{\hbar\partial k_\mu}$$

$$\langle u_{m\mathbf{k}}|v_\mu|u_{n\mathbf{k}}\rangle = \frac{1}{\hbar}(E_{n\mathbf{k}} - E_{m\mathbf{k}})\langle u_{m\mathbf{k}}|\frac{\partial}{\partial k_\mu}|u_{n\mathbf{k}}\rangle$$

$$\begin{aligned}\sigma_{xy} &= \frac{\langle j_x\rangle_E}{E} \\ &= -\frac{i\hbar e^2}{L^2}\sum_{n\neq m} f(E_n)\frac{\langle n|v_x|m\rangle\langle m|v_y|n\rangle - \langle n|v_y|m\rangle\langle m|v_x|n\rangle}{(E_n - E_m)^2}\end{aligned}$$

Hall conductivity

$$H\left|u_{n\mathbf{k}}\right\rangle = E_{n\mathbf{k}}\left|u_{n\mathbf{k}}\right\rangle$$

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$$\begin{aligned}\sigma_{xy} &= \frac{\langle j_x\rangle_E}{E} \\ &= -\frac{i\hbar e^2}{L^2}\sum_{n\neq m} f(E_n)\frac{\langle n|v_x|m\rangle\langle m|v_y|n\rangle - \langle n|v_y|m\rangle\langle m|v_x|n\rangle}{(E_n - E_m)^2}\end{aligned}$$

Hall conductivity

$$H|u_{n\boldsymbol{k}}\rangle = E_{n\boldsymbol{k}}|u_{n\boldsymbol{k}}\rangle$$

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$$\langle u_{m\mathbf{k}}|v_\mu|u_{n\mathbf{k}}\rangle = \frac{1}{\hbar}(E_{n\mathbf{k}} - E_{m\mathbf{k}})\langle u_{m\mathbf{k}}|\frac{\partial}{\partial k_\mu}|u_{n\mathbf{k}}\rangle$$

$$\begin{aligned}
\sigma_{xy} &= -\frac{ie^2}{\hbar L^2}\sum_{\mathbf{k}}\sum_{n\neq m} f(E_{n\mathbf{k}})\left(\langle\frac{\partial}{\partial k_x}u_{n\mathbf{k}}|u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|\frac{\partial}{\partial k_y}u_{n\mathbf{k}}\rangle - \langle\frac{\partial}{\partial k_y}u_{n\mathbf{k}}|u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|\frac{\partial}{\partial k_x}u_{n\mathbf{k}}\rangle\right) \\
&= -\frac{ie^2}{\hbar L^2}\sum_{\mathbf{k}}\sum_{n} f(E_{n\mathbf{k}})\left(\langle\frac{\partial}{\partial k_x}u_{n\mathbf{k}}|\frac{\partial}{\partial k_y}u_{n\mathbf{k}}\rangle - \langle\frac{\partial}{\partial k_y}u_{n\mathbf{k}}|\frac{\partial}{\partial k_x}u_{n\mathbf{k}}\rangle\right)
\end{aligned}$$

Hall conductivity

$$\mathbf{a}(\mathbf{k}) = -i\sum_n \langle u_{n\mathbf{k}} | \nabla_{\mathbf{k}} | u_{n\mathbf{k}} \rangle$$

$$\sigma_{xy} = \frac{e^2}{h}\int \frac{d^2k}{2\pi}\left(\frac{\partial a_y}{\partial k_x} - \frac{\partial a_x}{\partial k_y}\right)$$

$$= \frac{e^2}{h}\int \frac{d^2k}{2\pi}\left[\nabla_{\mathbf{k}} \times \mathbf{a}(\mathbf{k})\right]_z$$

$$\sigma_{xy} = -\frac{ie^2}{\hbar L^2}\sum_{\mathbf{k}}\sum_{n\neq m} f(E_{n\mathbf{k}})\left(\langle \frac{\partial}{\partial k_x}u_{n\mathbf{k}}|u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|\frac{\partial}{\partial k_y}u_{n\mathbf{k}}\rangle - \langle \frac{\partial}{\partial k_y}u_{n\mathbf{k}}|u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|\frac{\partial}{\partial k_x}u_{n\mathbf{k}}\rangle\right)$$

$$= -\frac{ie^2}{\hbar L^2}\sum_{\mathbf{k}}\sum_{n} f(E_{n\mathbf{k}})\left(\langle \frac{\partial}{\partial k_x}u_{n\mathbf{k}}|\frac{\partial}{\partial k_y}u_{n\mathbf{k}}\rangle - \langle \frac{\partial}{\partial k_y}u_{n\mathbf{k}}|\frac{\partial}{\partial k_x}u_{n\mathbf{k}}\rangle\right)$$

Hall conductivity

$$\boldsymbol{a}(\boldsymbol{k}) = -i\sum_n \langle u_{n\boldsymbol{k}} | \nabla_{\boldsymbol{k}} | u_{n\boldsymbol{k}} \rangle$$

$$\sigma_{xy} = \frac{e^2}{h}\int \frac{d^2k}{2\pi}\left(\frac{\partial a_y}{\partial k_x} - \frac{\partial a_x}{\partial k_y}\right)$$

$$= \frac{e^2}{h}\int \frac{d^2k}{2\pi}\left[\nabla_{\boldsymbol{k}} \times \boldsymbol{a}(\boldsymbol{k})\right]_z$$

TKNN formula

Thouless, Kohmoto, Nightingale, Nijs, PRL 49, 405 (1982).
Kohmoto, Ann. Phys. 160 355 (1985).

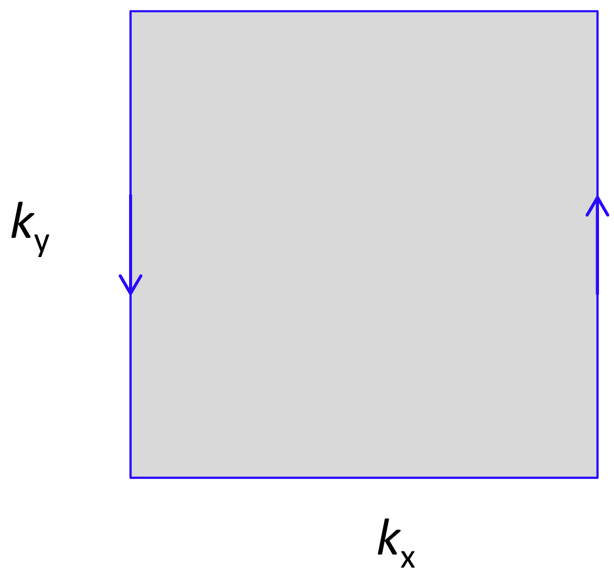
Hall conductivity

$$\boldsymbol{a}(\boldsymbol{k}) = -i\sum_n \langle u_{n\boldsymbol{k}} | \nabla_{\boldsymbol{k}} | u_{n\boldsymbol{k}} \rangle$$

$$\sigma_{xy} = \frac{e^2}{h}\int \frac{d^2k}{2\pi}\left(\frac{\partial a_y}{\partial k_x} - \frac{\partial a_x}{\partial k_y}\right)$$

$$= \frac{e^2}{h}\int \frac{d^2k}{2\pi}\left[\nabla_{\boldsymbol{k}} \times \boldsymbol{a}(\boldsymbol{k})\right]_z$$

$$\nu = \frac{1}{2\pi}\oint_{\partial BZ} d\boldsymbol{k}\cdot\boldsymbol{a}(\boldsymbol{k})$$



Hall conductivity

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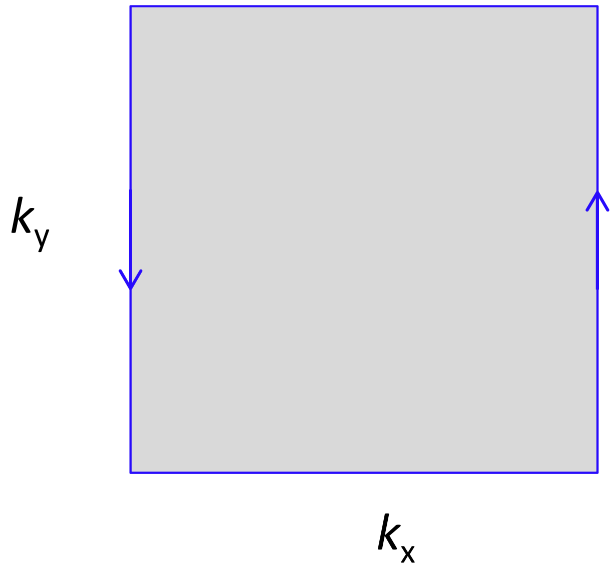
$$\nu = \frac{1}{2\pi}\oint_{\partial BZ} d\boldsymbol{k}\cdot\boldsymbol{a}(\boldsymbol{k})$$

= 0 !?



Hall conductivity

$$\boldsymbol{a}(\boldsymbol{k}) = -i\sum_n \langle u_{n\boldsymbol{k}} | \nabla_{\boldsymbol{k}} | u_{n\boldsymbol{k}} \rangle$$

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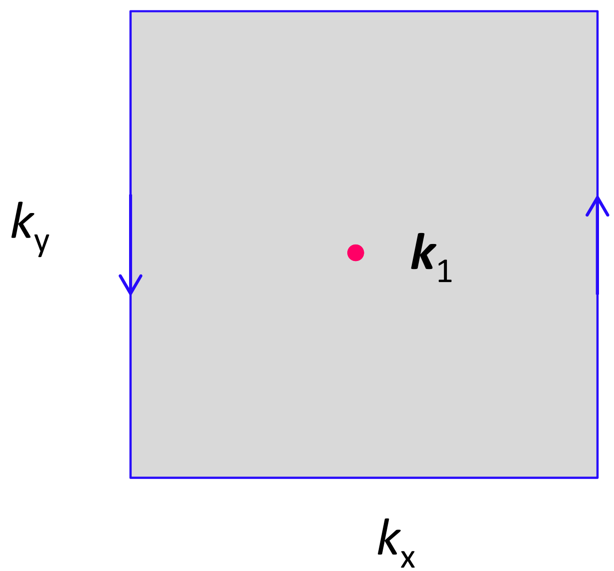
$$= \frac{e^2}{h}\int \frac{d^2k}{2\pi}\left[\nabla_{\boldsymbol{k}} \times \boldsymbol{a}(\boldsymbol{k})\right]_z$$

$$\nu = \frac{1}{2\pi}\oint_{\partial BZ} d\boldsymbol{k}\cdot\boldsymbol{a}(\boldsymbol{k})$$

= 0 !?

$$\boldsymbol{a}(\boldsymbol{k}_1) = \infty$$



Hall conductivity

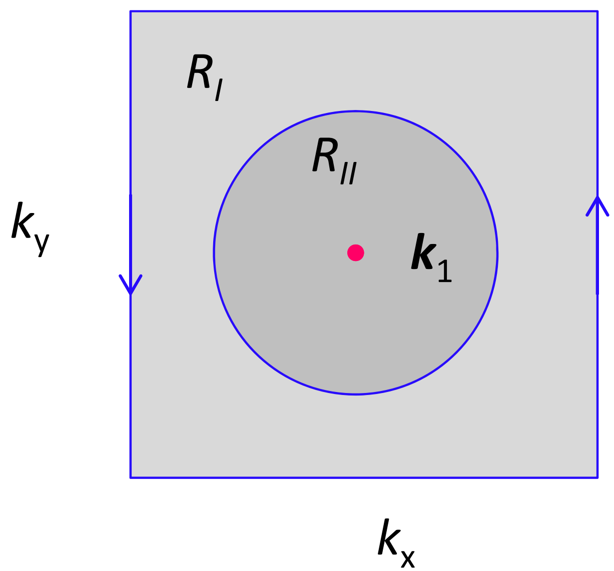
$$\boldsymbol{a}(\boldsymbol{k}) = -i\sum_n \langle u_{n\boldsymbol{k}} | \nabla_{\boldsymbol{k}} | u_{n\boldsymbol{k}} \rangle$$

$$\sigma_{xy} = \frac{e^2}{h}\int \frac{d^2k}{2\pi}\left(\frac{\partial a_y}{\partial k_x} - \frac{\partial a_x}{\partial k_y}\right)$$

$$= \frac{e^2}{h}\int \frac{d^2k}{2\pi}\left[\nabla_{\boldsymbol{k}} \times \boldsymbol{a}(\boldsymbol{k})\right]_z$$

$$\boldsymbol{a}^I(\boldsymbol{k}) = \boldsymbol{a}^{II}(\boldsymbol{k}) + \nabla_k \chi(\boldsymbol{k})$$

$$\nu = \frac{1}{2\pi}\oint_{\partial R_I} d\boldsymbol{k}\cdot\boldsymbol{a}^I(\boldsymbol{k}) + \frac{1}{2\pi}\oint_{\partial R_{II}} d\boldsymbol{k}\cdot\boldsymbol{a}^{II}(\boldsymbol{k})$$

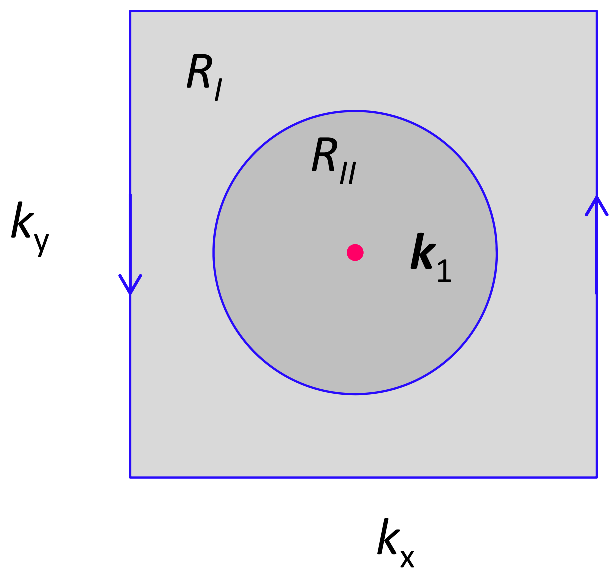


Hall conductivity

$$\nu = \frac{1}{2\pi}\oint_{\partial R_I} d\boldsymbol{k}\cdot\boldsymbol{a}^I(\boldsymbol{k}) + \frac{1}{2\pi}\oint_{\partial R_{II}} d\boldsymbol{k}\cdot\boldsymbol{a}^{II}(\boldsymbol{k})$$

$$= \frac{1}{2\pi}\oint_C d\boldsymbol{k}\cdot\left(\boldsymbol{a}^I(\boldsymbol{k}) - \boldsymbol{a}^{II}(\boldsymbol{k})\right)$$

$$\boldsymbol{a}^I(\boldsymbol{k}) = \boldsymbol{a}^{II}(\boldsymbol{k}) + \nabla_k \chi(\boldsymbol{k})$$



Hall conductivity

$$\nu = \frac{1}{2\pi}\oint_{\partial R_I} d\boldsymbol{k}\cdot\boldsymbol{a}^I(\boldsymbol{k}) + \frac{1}{2\pi}\oint_{\partial R_{II}} d\boldsymbol{k}\cdot\boldsymbol{a}^{II}(\boldsymbol{k})$$

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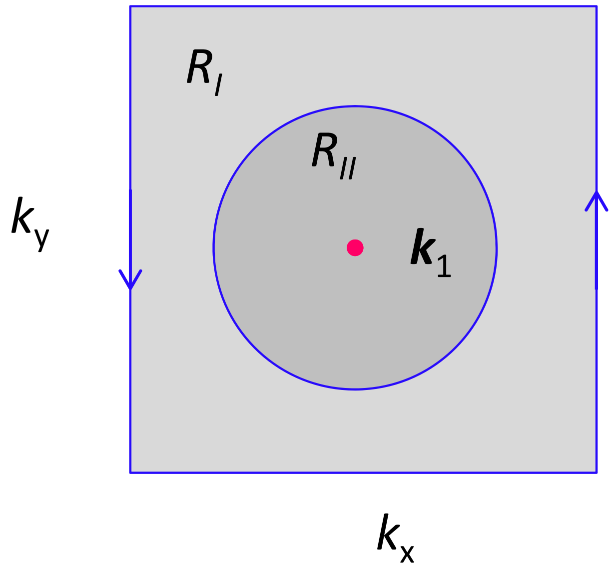

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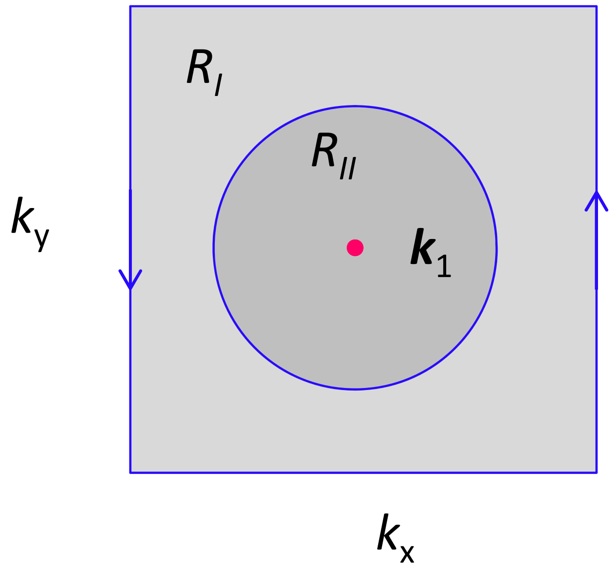
$$= \frac{1}{2\pi}\oint_C d\boldsymbol{k}\cdot\left(\boldsymbol{a}^I(\boldsymbol{k}) - \boldsymbol{a}^{II}(\boldsymbol{k})\right)$$

$$= \frac{1}{2\pi}\underbrace{\oint_C d\boldsymbol{k}\cdot\nabla_k\chi(\boldsymbol{k})}_{2\pi N}$$

$$\left|u^I_{n\boldsymbol{k}}\right\rangle = e^{i\chi(\boldsymbol{k})}\left|u^{II}_{n\boldsymbol{k}}\right\rangle$$

$$\boldsymbol{a}^I(\boldsymbol{k}) = \boldsymbol{a}^{II}(\boldsymbol{k}) + \nabla_k\chi(\boldsymbol{k})$$



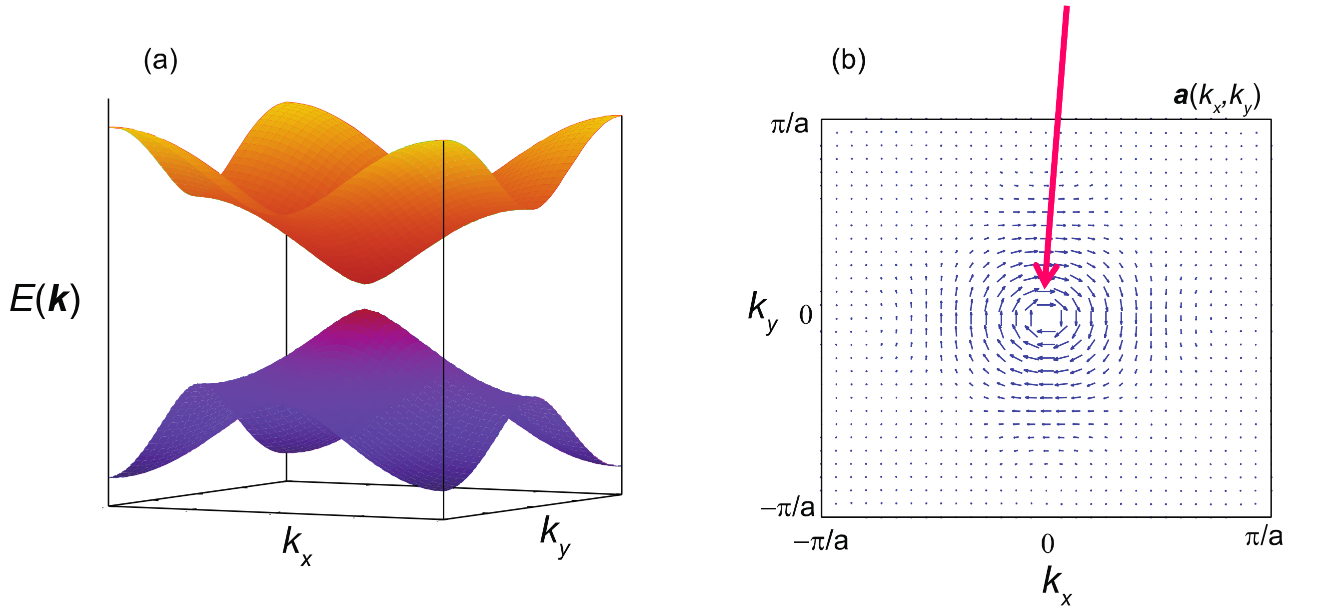
Hall conductivity

$$\nu = \frac{1}{2\pi}\oint_{\partial R_I} d\boldsymbol{k}\cdot\boldsymbol{a}^I(\boldsymbol{k}) + \frac{1}{2\pi}\oint_{\partial R_{II}} d\boldsymbol{k}\cdot\boldsymbol{a}^{II}(\boldsymbol{k})$$

$$= \frac{1}{2\pi}\oint_C d\boldsymbol{k}\cdot\left(\boldsymbol{a}^I(\boldsymbol{k}) - \boldsymbol{a}^{II}(\boldsymbol{k})\right)$$

$$= \frac{1}{2\pi}\oint_C d\boldsymbol{k}\cdot\nabla_k \chi(\boldsymbol{k})$$

“hole” $N = 1$

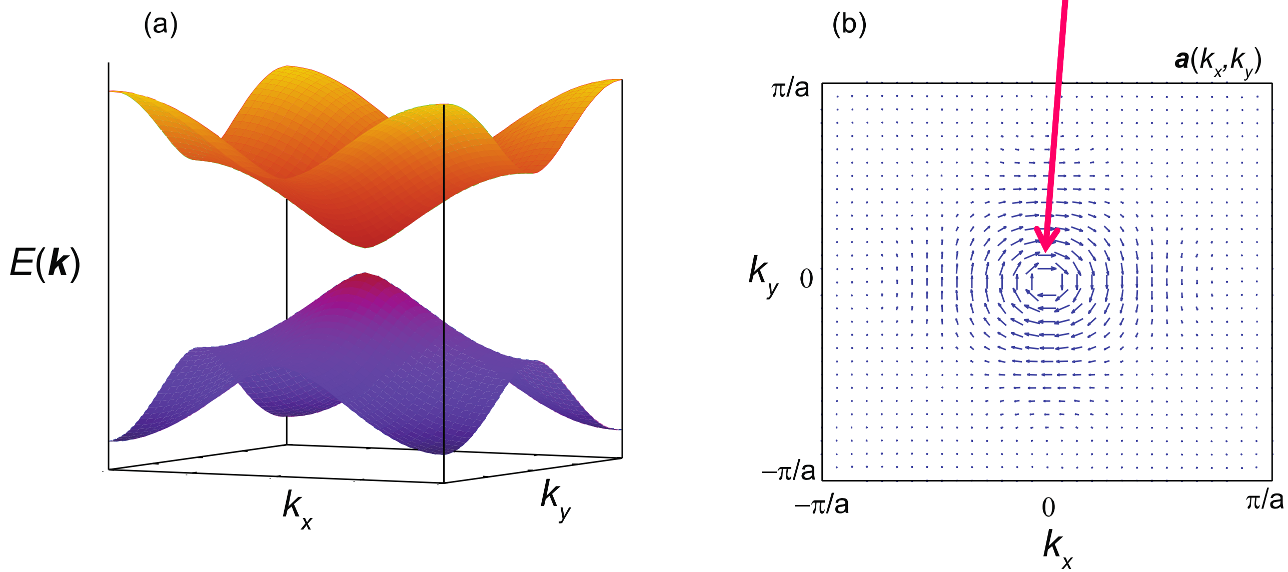


Hall conductivity

$$\begin{aligned}\mathcal{H}(\mathbf{k}) &= \epsilon(\mathbf{k}) + \boldsymbol{R}(k_x, k_y)\cdot\boldsymbol{\tau} \\ &= \begin{pmatrix} \epsilon(\mathbf{k}) + R_z(\mathbf{k}) & R_x(\mathbf{k}) - iR_y(\mathbf{k}) \\ R_x(\mathbf{k}) + iR_y(\mathbf{k}) & \epsilon(\mathbf{k}) - R_z(\mathbf{k}) \end{pmatrix}\end{aligned}$$

$$\boldsymbol{R}(k_x, k_y) = \begin{pmatrix} R_x(\mathbf{k}) \\ R_y(\mathbf{k}) \\ R_z(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} t\sin k_x a \\ t\sin k_y a \\ m + r\sum_{\mu=x,y}[1-\cos k_\mu a] \end{pmatrix}$$

“hole” $N = 1$



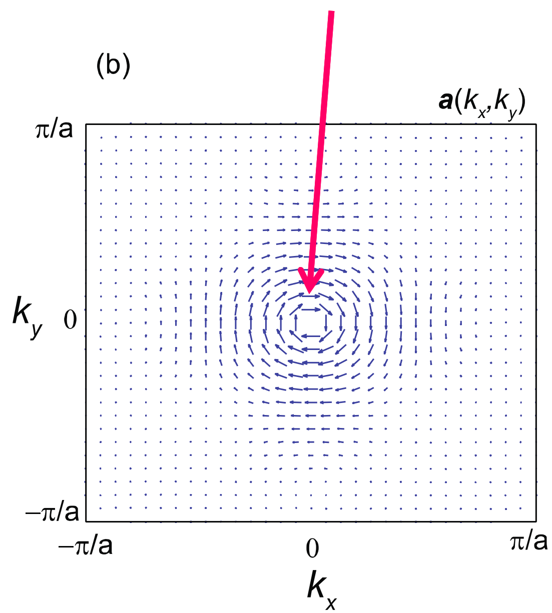
Hall conductivity

$$\mathcal{H}(\mathbf{k}) = \epsilon(\mathbf{k}) + \boldsymbol{R}(k_x, k_y)\cdot\boldsymbol{\tau}$$
$$= \begin{pmatrix} \epsilon(\mathbf{k}) + R_z(\mathbf{k}) & R_x(\mathbf{k}) - iR_y(\mathbf{k}) \\ R_x(\mathbf{k}) + iR_y(\mathbf{k}) & \epsilon(\mathbf{k}) - R_z(\mathbf{k}) \end{pmatrix}$$

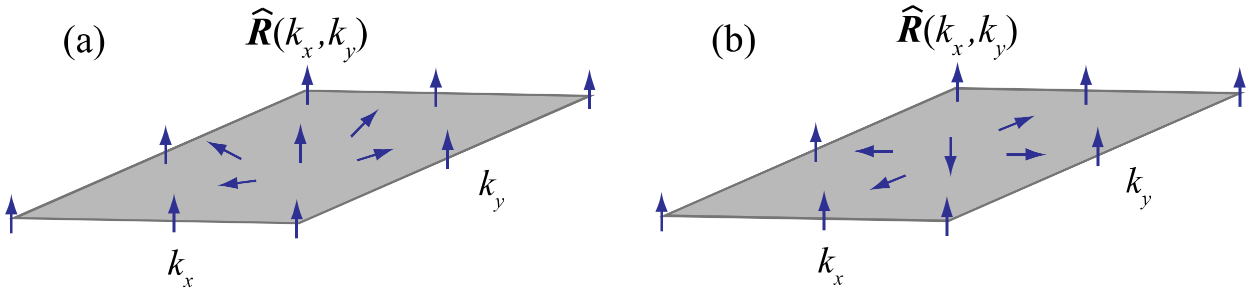
$$\boldsymbol{R}(k_x, k_y) = \begin{pmatrix} R_x(\mathbf{k}) \\ R_y(\mathbf{k}) \\ R_z(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} t\sin k_x a \\ t\sin k_y a \\ m + r\sum_{\mu=x,y}[1-\cos k_\mu a] \end{pmatrix}$$

"hole" $N = 1$

$$a_\mu^{\pm}(\mathbf{k}) = -i\langle \pm, \boldsymbol{R}(\mathbf{k})|\frac{\partial}{\partial k_\mu}|\pm, \boldsymbol{R}(\mathbf{k})\rangle$$
$$= -i\frac{\partial R_a(\mathbf{k})}{\partial k_\mu}\langle \pm, \boldsymbol{R}|\frac{\partial}{\partial R_a}|\pm, \boldsymbol{R}\rangle$$
$$= \frac{\partial R_a(\mathbf{k})}{\partial k_\mu}A_a^{\pm}(\boldsymbol{R}) \qquad (a = x, y, z)$$



Hall conductivity

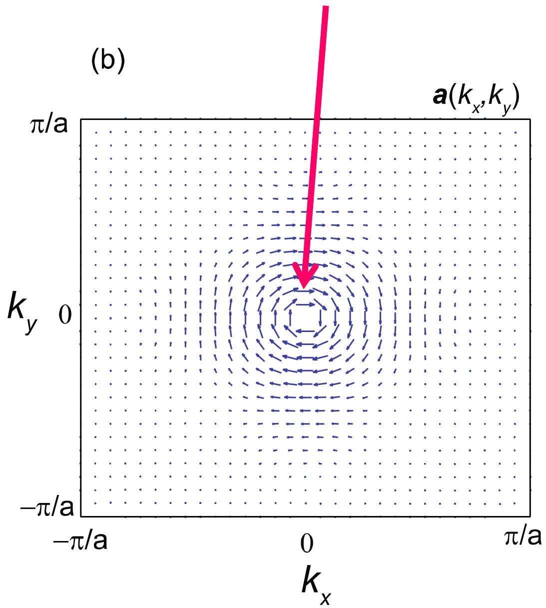


$m > 0$ $\quad$ $m < 0$

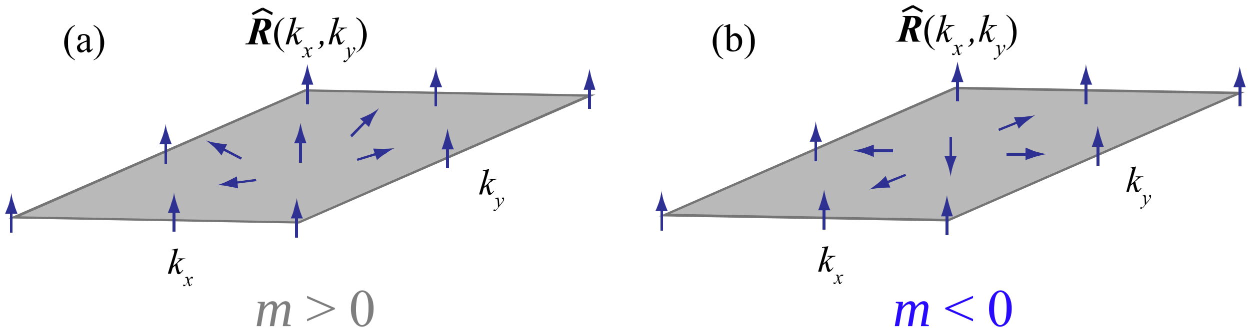
$$\boldsymbol{R}(k_x, k_y) = \begin{pmatrix} R_x(\mathbf{k}) \\ R_y(\mathbf{k}) \\ R_z(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} t \sin k_x a \\ t \sin k_y a \\ m + r \sum_{\mu=x,y}[1 - \cos k_\mu a] \end{pmatrix}$$

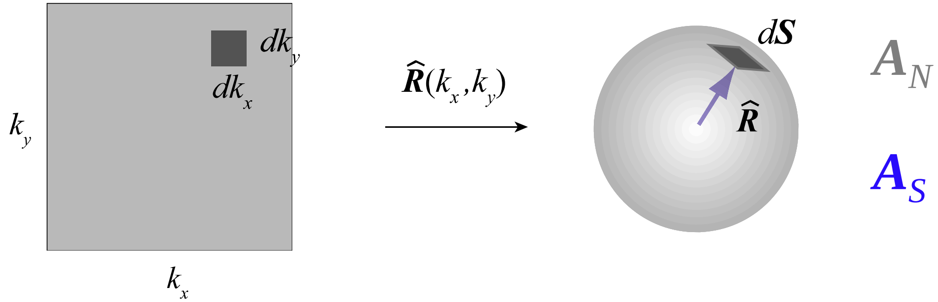
"hole" $N = 1$

$$\begin{aligned} a_\mu^{\pm}(\mathbf{k}) &= -i\langle \pm, \boldsymbol{R}(\mathbf{k})| \frac{\partial}{\partial k_\mu} |\pm, \boldsymbol{R}(\mathbf{k})\rangle \\ &= -i \frac{\partial R_a(\mathbf{k})}{\partial k_\mu} \langle \pm, \boldsymbol{R}| \frac{\partial}{\partial R_a} |\pm, \boldsymbol{R}\rangle \\ &= \frac{\partial R_a(\mathbf{k})}{\partial k_\mu} A_a^{\pm}(\boldsymbol{R}) \qquad (a = x, y, z) \end{aligned}$$

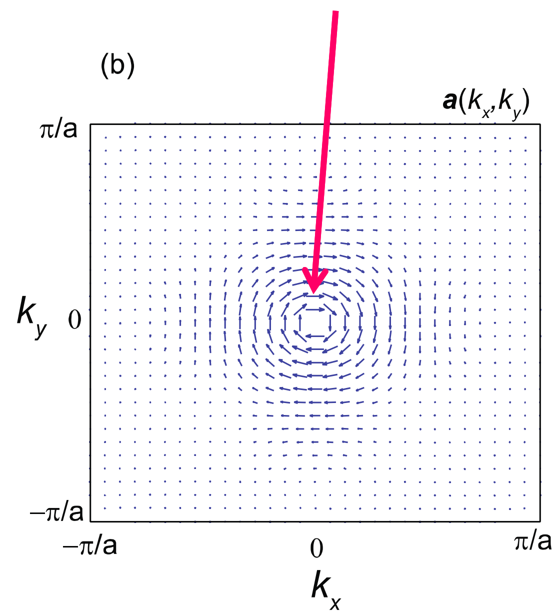


Hall conductivity



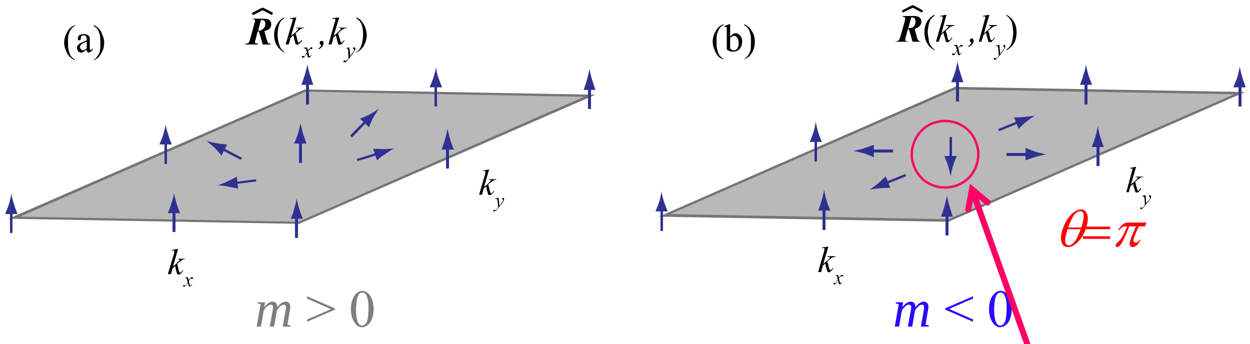


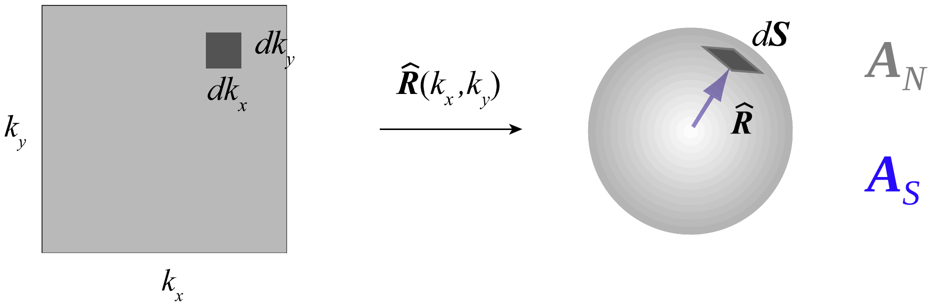
"hole" $N = 1$



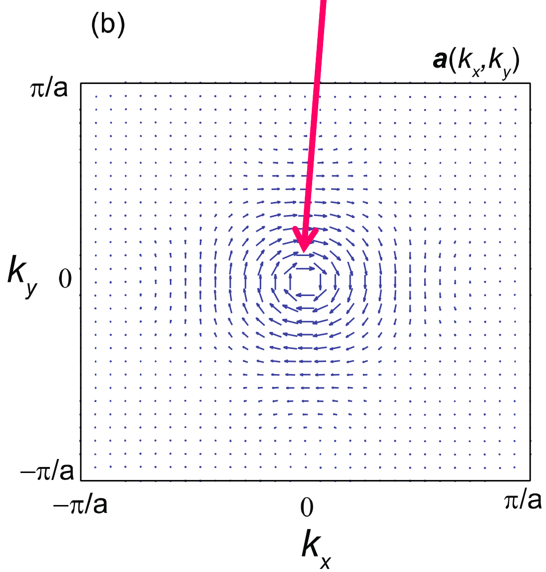
$$
\begin{aligned}
a_\mu^{\pm}(\mathbf{k}) &= -i\langle \pm, \boldsymbol{R}(\mathbf{k})|\frac{\partial}{\partial k_\mu}|\pm, \boldsymbol{R}(\mathbf{k})\rangle \\
&= -i\frac{\partial R_a(\mathbf{k})}{\partial k_\mu}\langle \pm, \boldsymbol{R}|\frac{\partial}{\partial R_a}|\pm, \boldsymbol{R}\rangle \\
&= \frac{\partial R_a(\mathbf{k})}{\partial k_\mu}A_a^{\pm}(\boldsymbol{R}) \qquad (a = x, y, z)
\end{aligned}
$$

Hall conductivity





"hole" $N=1$



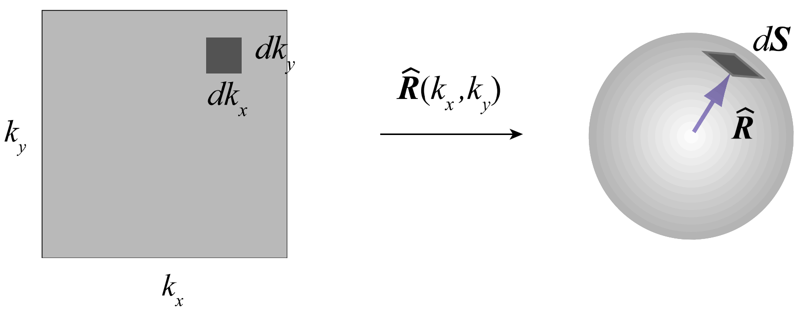
$$
\begin{aligned}
a_\mu^{\pm}(\mathbf{k}) &= -i\langle \pm, \boldsymbol{R}(\mathbf{k})|\frac{\partial}{\partial k_\mu}|\pm, \boldsymbol{R}(\mathbf{k})\rangle \\
&= -i\frac{\partial R_a(\mathbf{k})}{\partial k_\mu}\langle \pm, \boldsymbol{R}|\frac{\partial}{\partial R_a}|\pm, \boldsymbol{R}\rangle \\
&= \frac{\partial R_a(\mathbf{k})}{\partial k_\mu}A_a^{\pm}(\boldsymbol{R}) \qquad (a = x, y, z)
\end{aligned}
$$

$$
\boldsymbol{A}_N(\boldsymbol{R}) = \frac{1-\cos\theta}{2R\sin\theta}\boldsymbol{e}_\phi
$$

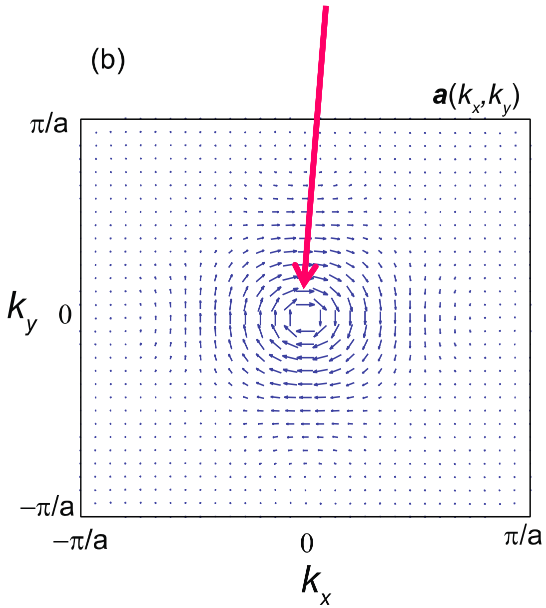
Hall conductivity

$$\sigma_{xy} = -\frac{e^2}{h}\int_{\rm BZ}\frac{d^2\mathbf{k}}{4\pi}\,\hat{\boldsymbol{R}}\cdot\left(\frac{\partial\hat{\boldsymbol{R}}}{\partial k_x}\times\frac{\partial\hat{\boldsymbol{R}}}{\partial k_y}\right)$$

$$d\boldsymbol{S} = \left(\frac{\partial\hat{\boldsymbol{R}}}{\partial k_x}dk_x\right)\times\left(\frac{\partial\hat{\boldsymbol{R}}}{\partial k_y}dk_y\right)$$



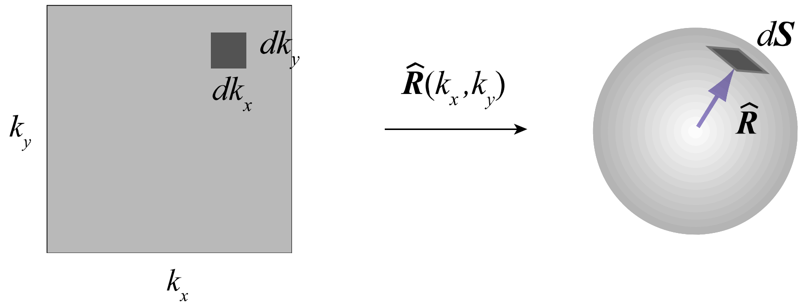
“hole” $N = 1$



$$\begin{aligned}
a_\mu^{\pm}(\mathbf{k}) &= -i\langle\pm,\boldsymbol{R}(\mathbf{k})|\frac{\partial}{\partial k_\mu}|\pm,\boldsymbol{R}(\mathbf{k})\rangle \\
&= -i\frac{\partial R_a(\mathbf{k})}{\partial k_\mu}\langle\pm,\boldsymbol{R}|\frac{\partial}{\partial R_a}|\pm,\boldsymbol{R}\rangle \\
&= \frac{\partial R_a(\mathbf{k})}{\partial k_\mu}A_a^{\pm}(\boldsymbol{R}) \qquad (a = x, y, z)
\end{aligned}$$

Hall conductivity

$$\sigma_{xy} = -\frac{e^2}{h}\int_{\mathrm{BZ}} \frac{d^2\mathbf{k}}{4\pi}\ \hat{\boldsymbol{R}}\cdot\Big(\frac{\partial \hat{\boldsymbol{R}}}{\partial k_x}\times\frac{\partial \hat{\boldsymbol{R}}}{\partial k_y}\Big)$$



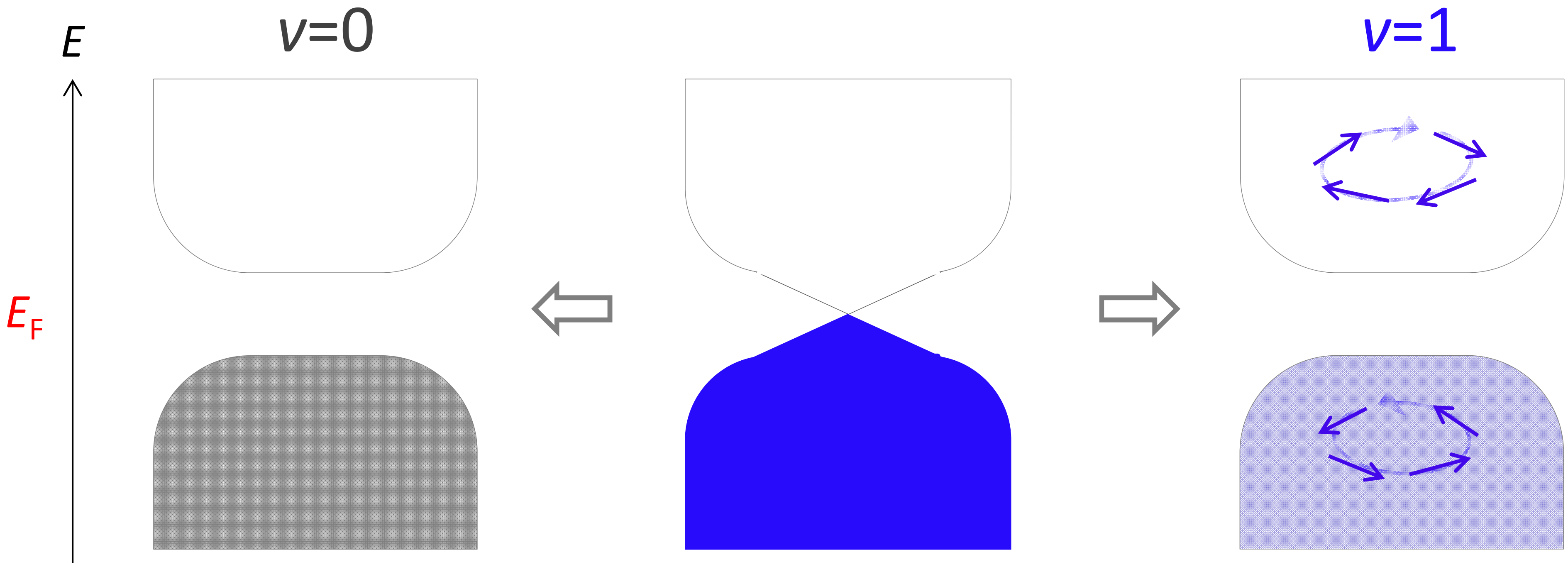
$$d\boldsymbol{S} = \Big(\frac{\partial \hat{\boldsymbol{R}}}{\partial k_x}dk_x\Big)\times\Big(\frac{\partial \hat{\boldsymbol{R}}}{\partial k_y}dk_y\Big)$$

$$\begin{aligned}
a^{\pm}_{\mu}(\mathbf{k}) &= -i\langle \pm, \boldsymbol{R}(\mathbf{k})|\frac{\partial}{\partial k_\mu}|\pm, \boldsymbol{R}(\mathbf{k})\rangle \\
&= -i\frac{\partial R_a(\mathbf{k})}{\partial k_\mu}\langle \pm, \boldsymbol{R}|\frac{\partial}{\partial R_a}|\pm, \boldsymbol{R}\rangle \\
&= \frac{\partial R_a(\mathbf{k})}{\partial k_\mu}A^{\pm}_a(\boldsymbol{R}) \qquad (a = x, y, z)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial a^{\pm}_y}{\partial k_x} - \frac{\partial a^{\pm}_x}{\partial k_y} \\
&= \frac{\partial}{\partial k_x}\Big(\frac{\partial R_b}{\partial k_y}A^{\pm}_b(\boldsymbol{R})\Big) - \frac{\partial}{\partial k_y}\Big(\frac{\partial R_a}{\partial k_x}A^{\pm}_a(\boldsymbol{R})\Big) \\
&= \frac{\partial^2 R_b}{\partial k_x \partial k_y}A^{\pm}_b + \frac{\partial R_b}{\partial k_y}\frac{\partial R_a}{\partial k_x}\frac{\partial A^{\pm}_b}{\partial R_a} - \frac{\partial^2 R_a}{\partial k_y \partial k_x}A^{\pm}_a - \frac{\partial R_a}{\partial k_x}\frac{\partial R_b}{\partial k_y}\frac{\partial A^{\pm}_a}{\partial R_b} \\
&= \frac{\partial R_a}{\partial k_x}\frac{\partial R_b}{\partial k_y}\Big(\frac{\partial A^{\pm}_b}{\partial R_a} - \frac{\partial A^{\pm}_a}{\partial R_b}\Big) \\
&= \frac{\partial R_a}{\partial k_x}\frac{\partial R_b}{\partial k_y}\epsilon_{abc}B^{\pm}_c = \frac{\partial R_a}{\partial k_x}\frac{\partial R_b}{\partial k_y}\epsilon_{abc}\Big(\pm\frac{1}{2}\frac{R_c}{R^3}\Big) \\
&= \pm\frac{1}{2R^3}\boldsymbol{R}\cdot\Big(\frac{\partial \boldsymbol{R}}{\partial k_x}\times\frac{\partial \boldsymbol{R}}{\partial k_y}\Big)
\end{aligned}$$

How topology changes?

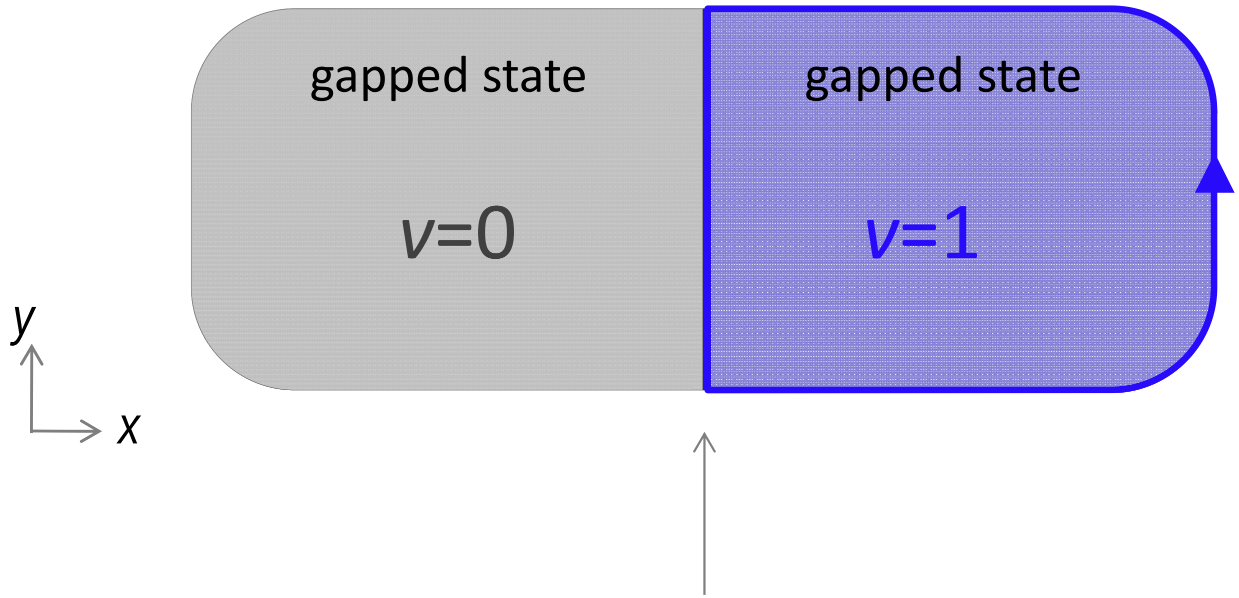
Transition between different topological phases
for example $\nu = \mathbf{0}$ and $\mathbf{1}$



The gap closes
at the transition point

Edge states

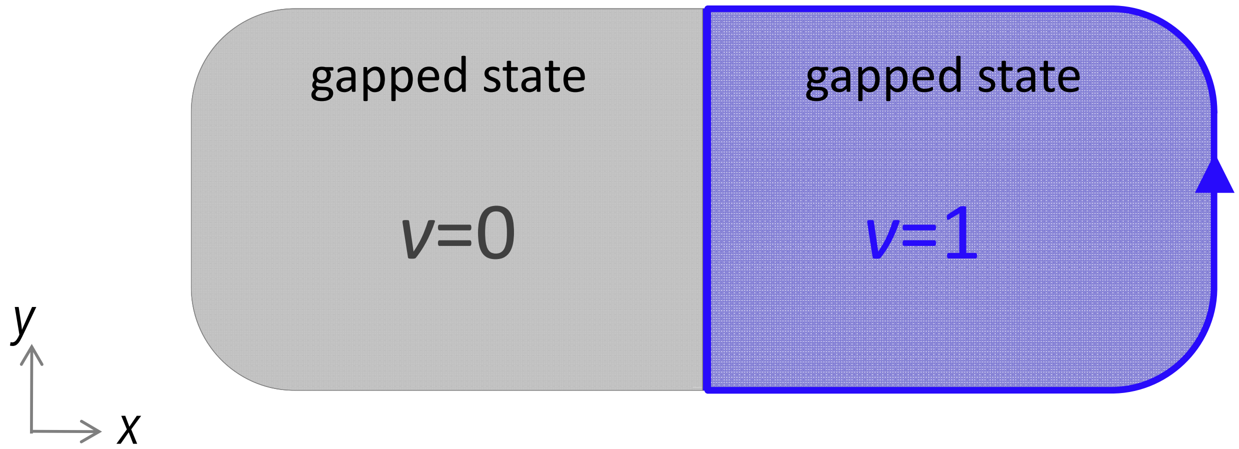
Two topologically distinct insulators
attached with each other

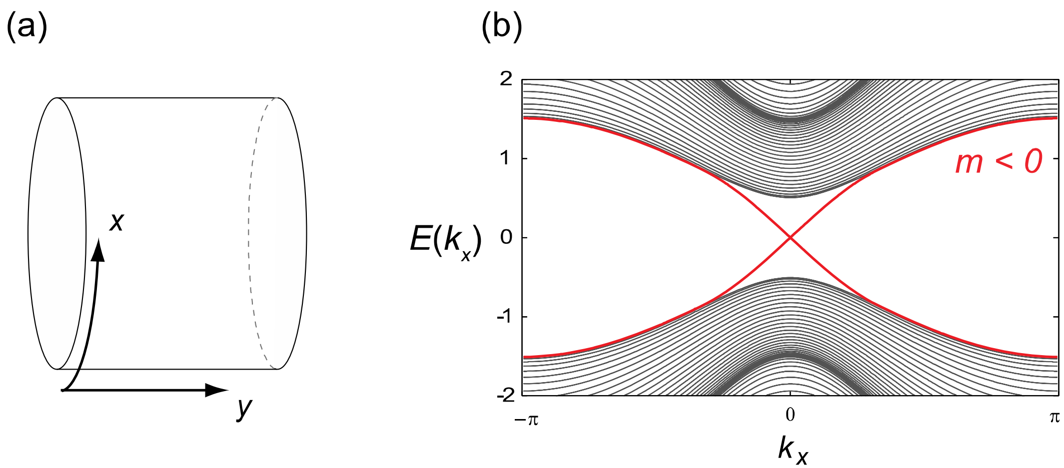


The gap closes
at the boundary
= gapless edge modes

Edge states

Two topologically distinct insulators
attached with each other





Lecture at JAEA 1/23/2017 RIKEN

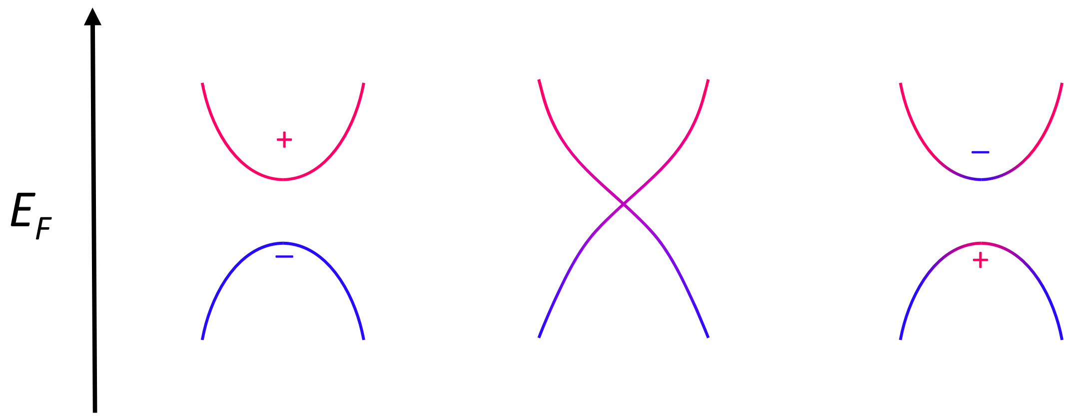
Introduction to Topological Insulators

Kentaro Nomura (IMR, Tohoku)

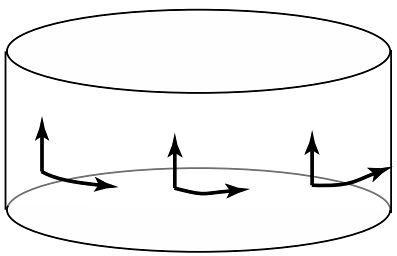
outline

- Quantum Hall effect
- Z2 topological insulators
- Electromagnetic responses

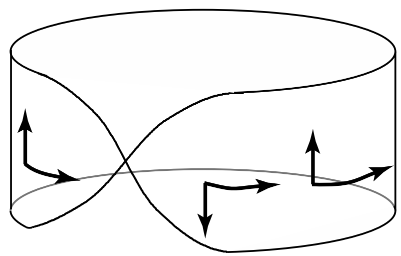
What is topology insulators?



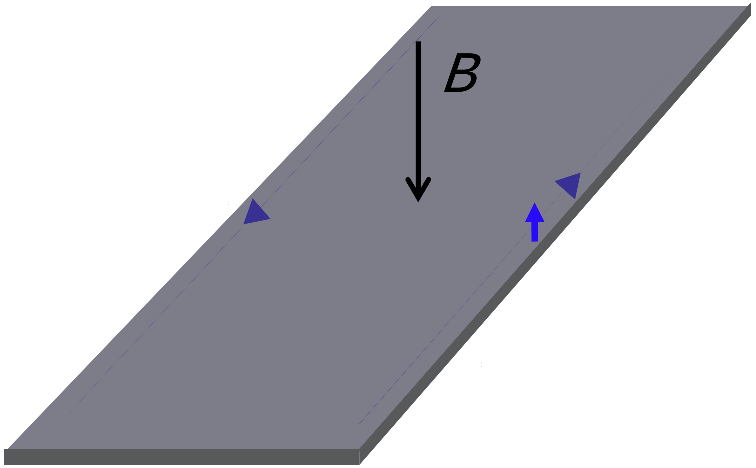
Trivial insulator



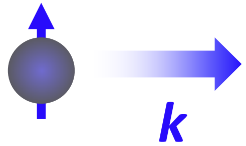
topological insulator



Basic idea

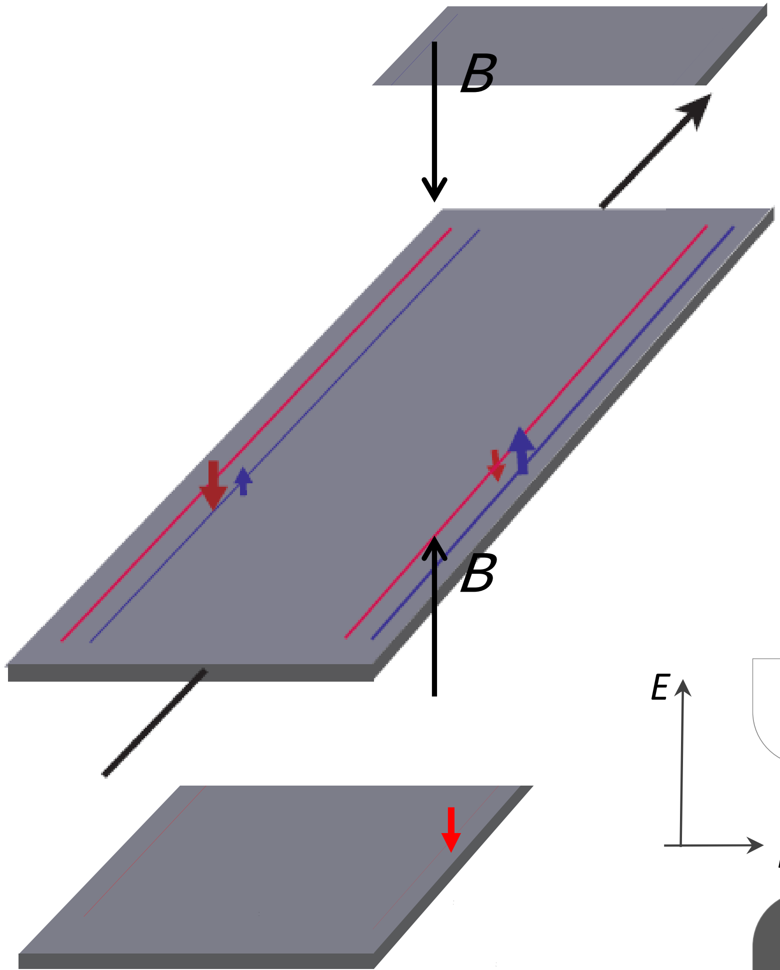


QHE up spin



Quantum Hall Effect (QHE) is realized when time-reversal symmetry is broken

Basic idea



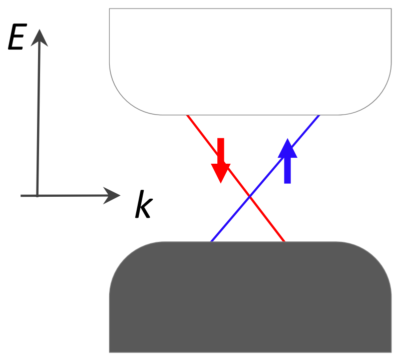
Quantum spin Hall effect (QSHE)

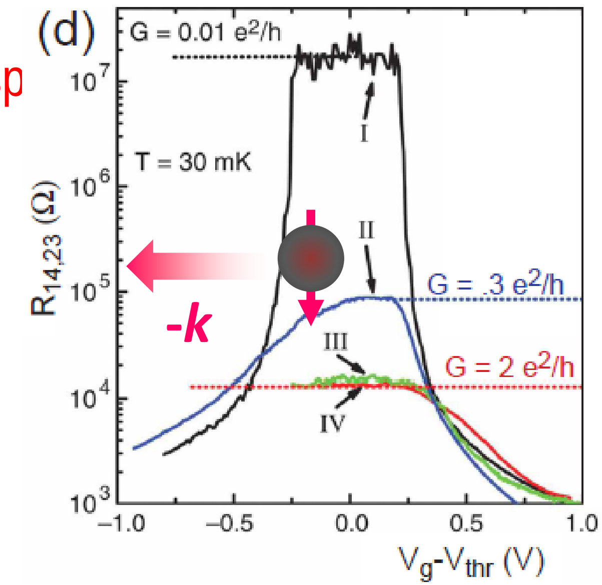
QHE up spin

Theory
- Kane-Mele (2005)
- Bernevig-Zhang (2006)
- Bernevig-Hughes-Zhang (2006)

Experiment
- Molemkamp goup (2007)   HgTe QW

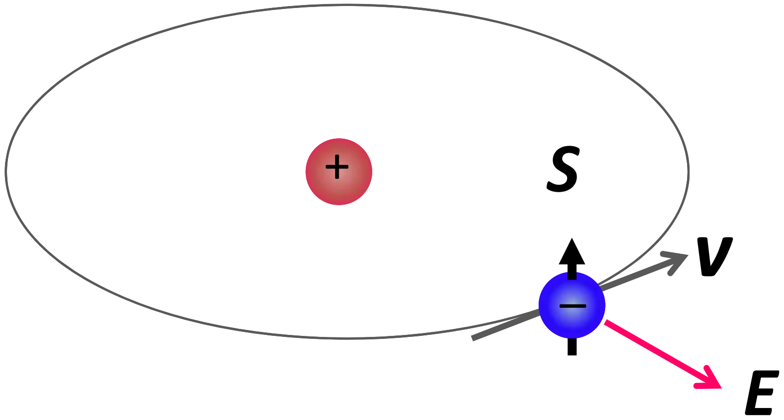
QHE down sp





Spin–Orbit Coupling (SOC)

SOC in an atom



Moving electrons feel
an effective magnetic field

$$\boldsymbol{B}_{eff} = -\frac{\boldsymbol{v} \times \boldsymbol{E}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$H_{so} = -g\mu_B \boldsymbol{S} \cdot \boldsymbol{B}_{eff}$$

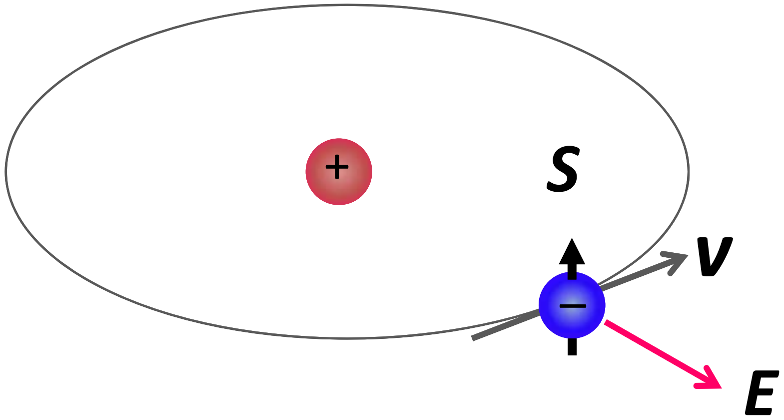
$$= \frac{-e\hbar}{2m^2c^2} \boldsymbol{p} \times \boldsymbol{E} \cdot \boldsymbol{S}$$

$$= \lambda\, \boldsymbol{r} \times \boldsymbol{p} \cdot \boldsymbol{S}$$

$$H_{so} = \lambda\, \boldsymbol{L} \cdot \boldsymbol{S}$$

Spin–Orbit Coupling (SOC)

SOC in an atom



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$$= \frac{-e\hbar}{2m^2c^2}\boldsymbol{p}\times\boldsymbol{E}\cdot\boldsymbol{S}$$

$$= \lambda\, \boldsymbol{r}\times\boldsymbol{p}\cdot\boldsymbol{S}$$

Analogy with a magnetic field

$$H = \frac{(\boldsymbol{p}-e\boldsymbol{A})^2}{2m}$$

$$H_{B-orbital} = \frac{-e}{m}\boldsymbol{p}\cdot\boldsymbol{A}$$

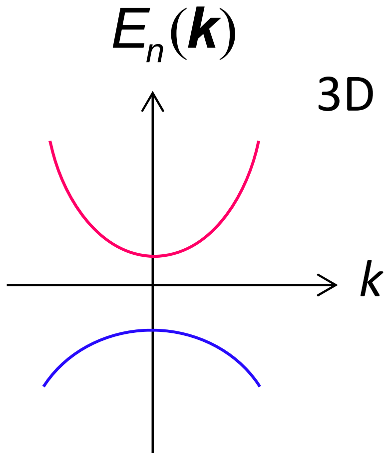
$$\boldsymbol{A} = \frac{1}{2}\boldsymbol{B}\times\boldsymbol{r}$$

$$= \frac{-e}{2m}\boldsymbol{r}\times\boldsymbol{p}\cdot\boldsymbol{B}$$

Insulator (Semiconductor)

3D bulk

s−band



p−band

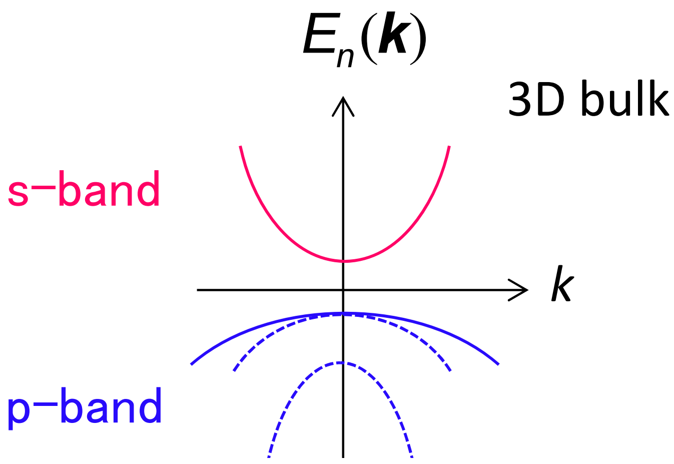
						2 He
	5 B	6 C	7 N	8 O	9 F	10 Ne
	13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
112 Cn	113 Uut	114 Fl	115 Uup	116 Lv	117 Uus	118 Uuo

$L = 1$

$L_z = +1, 0, -1$

3−fold degeneracy
(without spin)

Insulator (Semiconductor)



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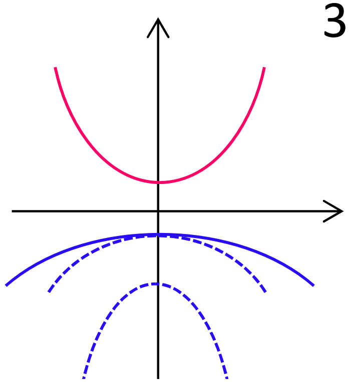
$L = 1$

$L_z = +1, 0, -1$

~~3−fold degeneracy~~
~~(without spin)~~

Insulator (Semiconductor)



J* = *L* + *S

j=3/2

j_z=+3/2, −3/2 (heavy hole band)

j_z=+1/2, −1/2 (light hole band)

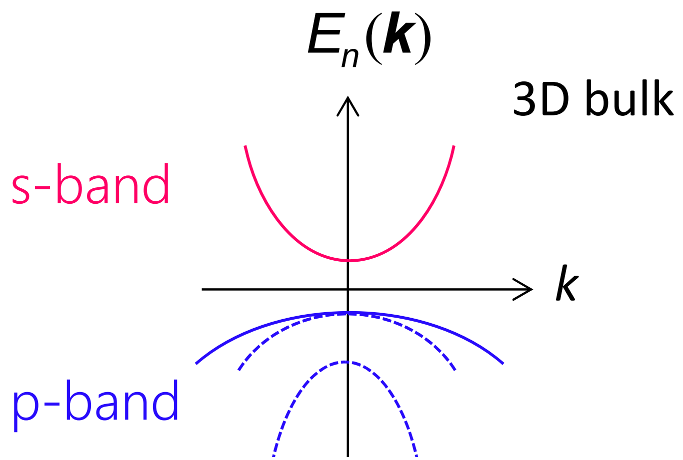
j=1/2

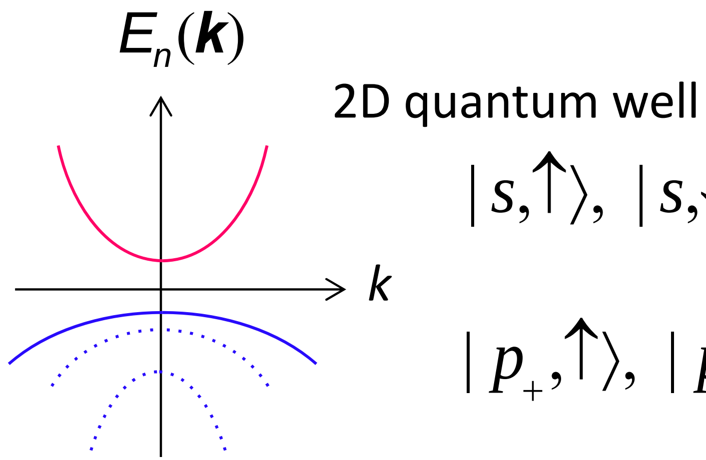
j_z=+1/2, −1/2 (split off band)

$$H_{so} = \lambda \boldsymbol{L} \cdot \boldsymbol{S}$$

$$= \frac{\lambda}{2}\left(\boldsymbol{J}^2 - \boldsymbol{L}^2 - \boldsymbol{S}^2\right)$$

2D (quantum well)





$|s,\uparrow\rangle,\ |s,\downarrow\rangle$

$|p_+,\uparrow\rangle,\ |p_-,\downarrow\rangle$

J* = *L* + *S

j=3/2

j_z=+3/2, -3/2 (heavy hole band)

j_z=+1/2, -1/2 (light hole band)

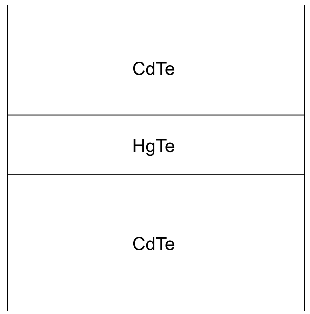
j=1/2

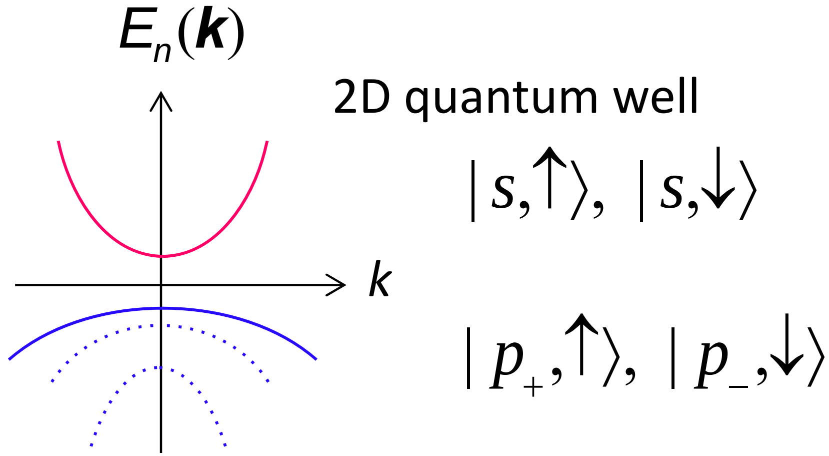
j_z=+1/2, -1/2 (split off band)

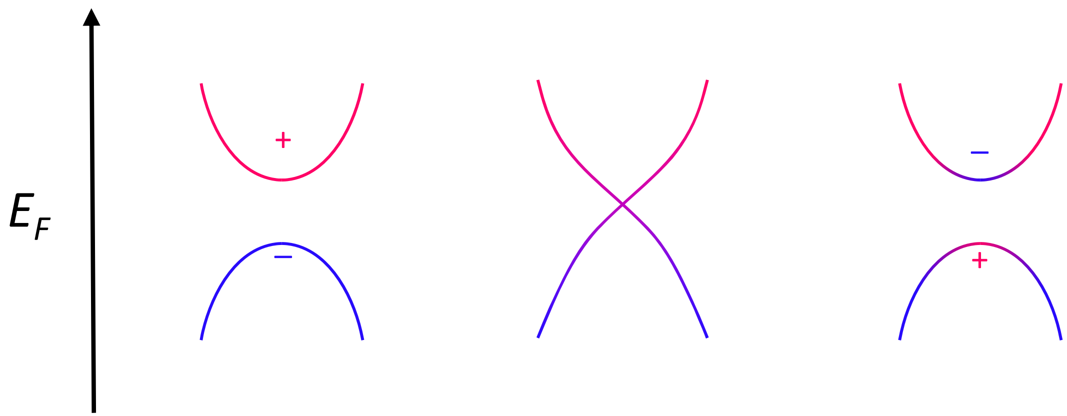
$$H_{so} = \lambda\, \boldsymbol{L}\cdot\boldsymbol{S}$$

$$= \frac{\lambda}{2}\left(\boldsymbol{J}^2 - \boldsymbol{L}^2 - \boldsymbol{S}^2\right)$$

2dTI in HgTe/CdTe quantum well



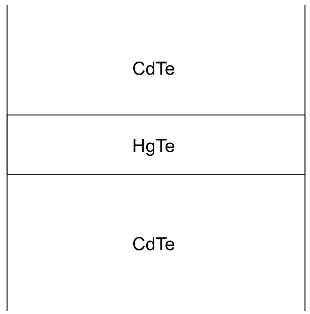


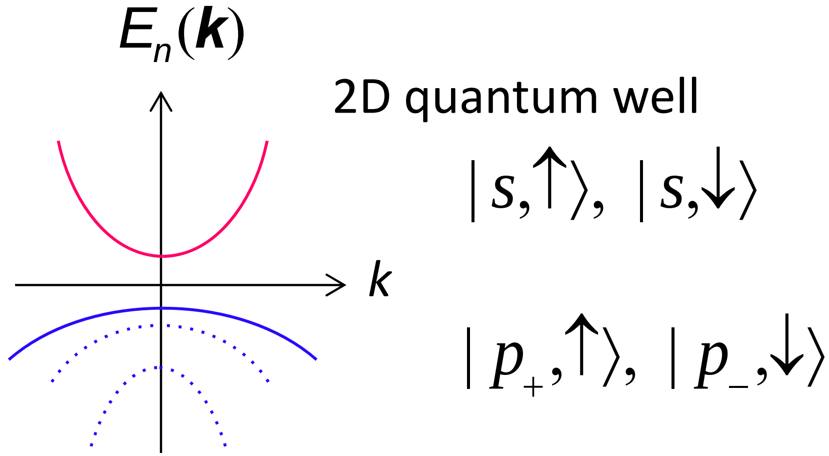


Trivial insulator

topological insulator

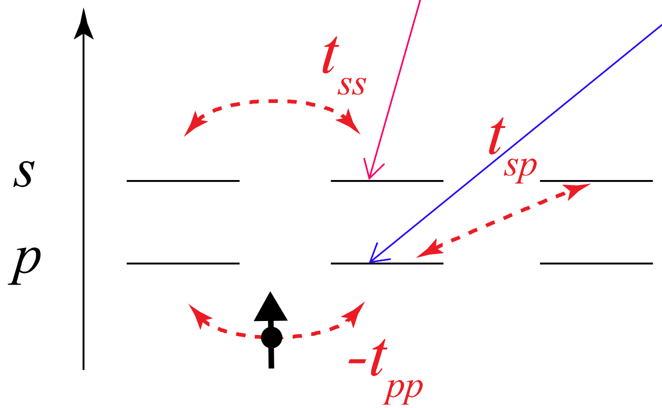
2dTI in HgTe/CdTe quantum well

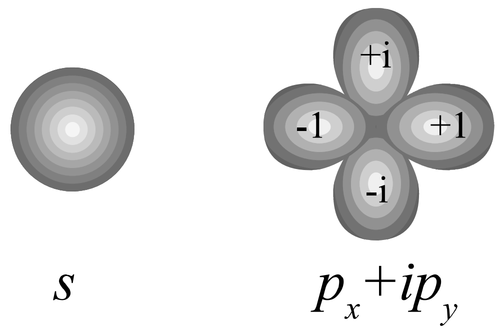




$$H_0 = \sum_{\boldsymbol{R},\sigma_z} \left(\epsilon_s |\boldsymbol{R}, s, \sigma_z\rangle\langle \boldsymbol{R}, s, \sigma_z| + \epsilon_p |\boldsymbol{R}, p_{\sigma_z}, \sigma_z\rangle\langle \boldsymbol{R}, p_{\sigma_z}, \sigma_z| \right)$$

Bernevig, Hughes, Zhang (2006)



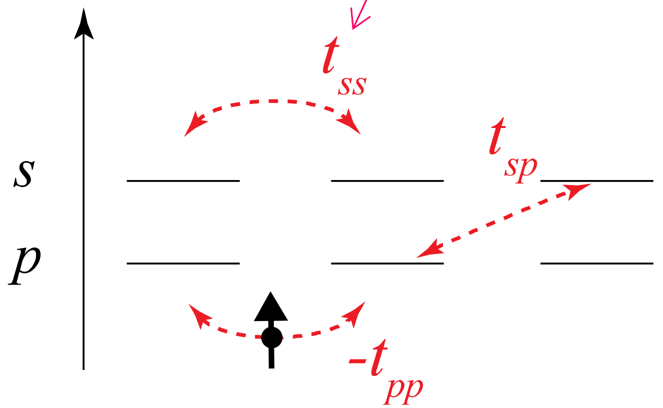


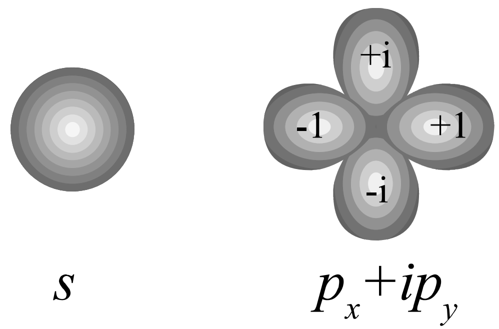
2dTI in HgTe/CdTe quantum well

$$H_t = -\sum_{\boldsymbol{R},\sigma_z} \sum_{\mu=\pm x,\pm y} t_{ss} |\boldsymbol{R}+\mathbf{e}_\mu, s, \sigma_z\rangle\langle \boldsymbol{R}, s, \sigma_z|$$

$$H_0 = \sum_{\boldsymbol{R},\sigma_z} \left(\epsilon_s |\boldsymbol{R}, s, \sigma_z\rangle\langle \boldsymbol{R}, s, \sigma_z| + \epsilon_p |\boldsymbol{R}, p_{\sigma_z}, \sigma_z\rangle\langle \boldsymbol{R}, p_{\sigma_z}, \sigma_z| \right)$$

Bernevig, Hughes, Zhang (2006)



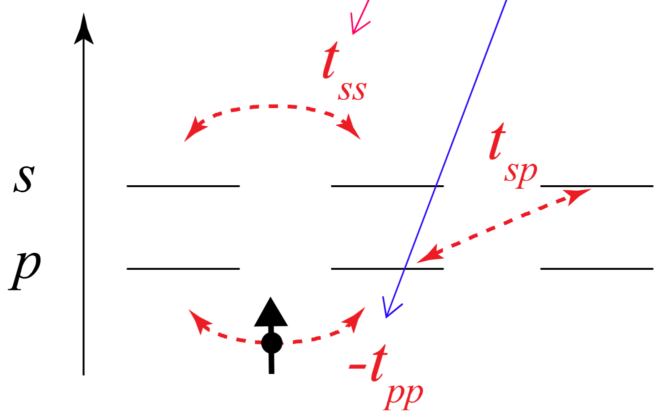


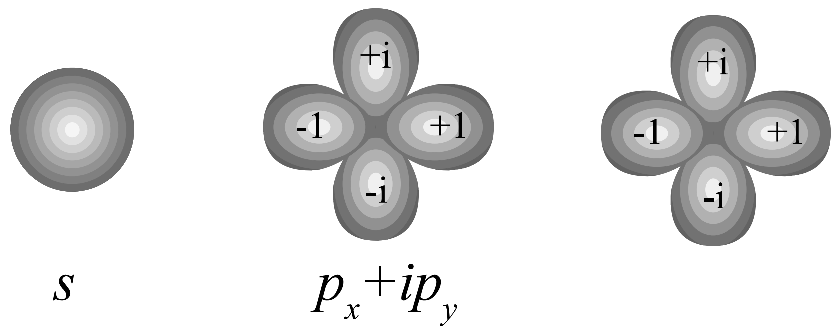
2dTI in HgTe/CdTe quantum well

$$H_t = -\sum_{\boldsymbol{R},\sigma_z}\sum_{\mu=\pm x,\pm y}\Big(t_{ss}|\boldsymbol{R}+\mathbf{e}_\mu, s, \sigma_z\rangle\langle\boldsymbol{R}, s, \sigma_z| - t_{pp}|\boldsymbol{R}+\mathbf{e}_\mu, p_{\sigma_z}, \sigma_z\rangle\langle\boldsymbol{R}, p_{\sigma_z}, \sigma_z|$$

$$H_0 = \sum_{\boldsymbol{R},\sigma_z}\Big(\epsilon_s|\boldsymbol{R}, s, \sigma_z\rangle\langle\boldsymbol{R}, s, \sigma_z| + \epsilon_p|\boldsymbol{R}, p_{\sigma_z}, \sigma_z\rangle\langle\boldsymbol{R}, p_{\sigma_z}, \sigma_z|\Big)$$

Bernevig, Hughes, Zhang (2006)



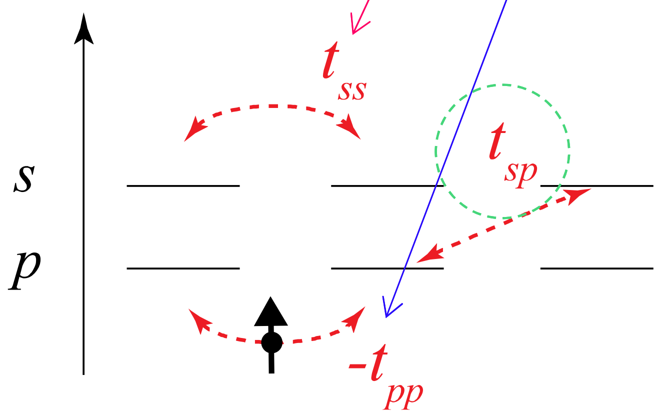


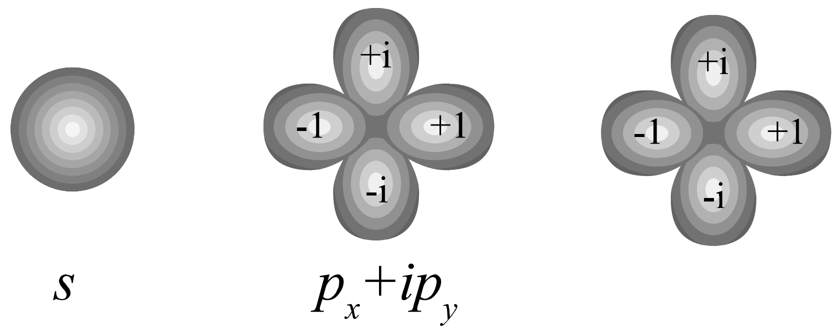
2dTI in HgTe/CdTe quantum well

$$
\begin{aligned}
H_t &= -\sum_{\boldsymbol{R},\sigma_z}\sum_{\mu=\pm x,\pm y}\Big(t_{ss}|\boldsymbol{R}+\mathbf{e}_\mu, s,\sigma_z\rangle\langle \boldsymbol{R}, s,\sigma_z| \\
&\qquad -t_{pp}|\boldsymbol{R}+\mathbf{e}_\mu, p_{\sigma_z},\sigma_z\rangle\langle \boldsymbol{R}, p_{\sigma_z},\sigma_z| \\
&\qquad +t_{sp}e^{i\theta_\mu\sigma_z}|\boldsymbol{R}+\mathbf{e}_\mu, s,\sigma_z\rangle\langle \boldsymbol{R}, p_{\sigma_z},\sigma_z| \\
&\qquad +t_{sp}e^{-i\theta_\mu\sigma_z}|\boldsymbol{R}+\mathbf{e}_\mu, p_{\sigma_z},\sigma_z\rangle\langle \boldsymbol{R}, s,\sigma_z| \Big)
\end{aligned}
$$

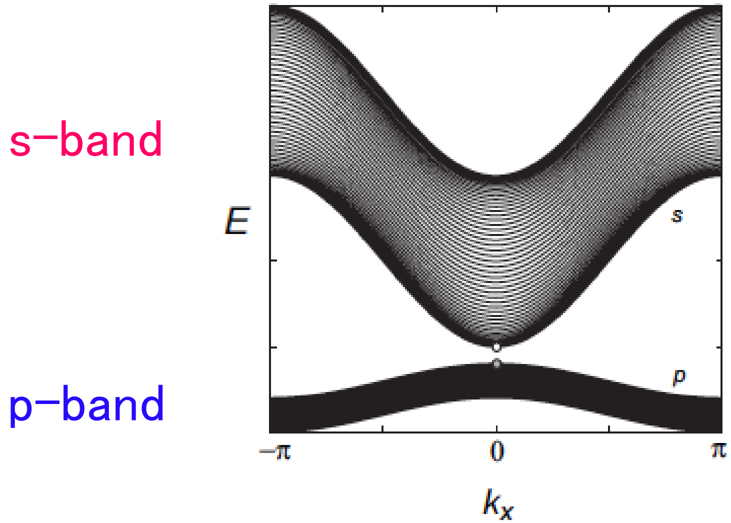
$$
H_0 = \sum_{\boldsymbol{R},\sigma_z}\Big(\epsilon_s|\boldsymbol{R}, s,\sigma_z\rangle\langle \boldsymbol{R}, s,\sigma_z| \;+\; \epsilon_p|\boldsymbol{R}, p_{\sigma_z},\sigma_z\rangle\langle \boldsymbol{R}, p_{\sigma_z},\sigma_z|\Big)
$$

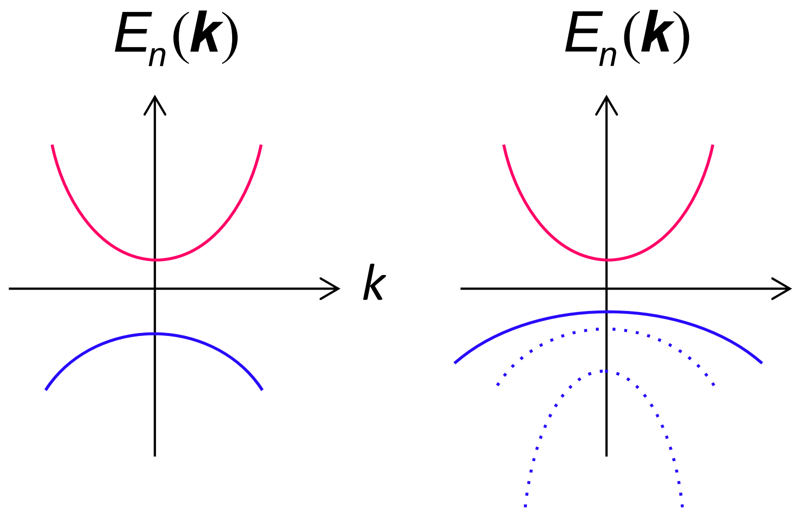
Bernevig, Hughes, Zhang (2006)

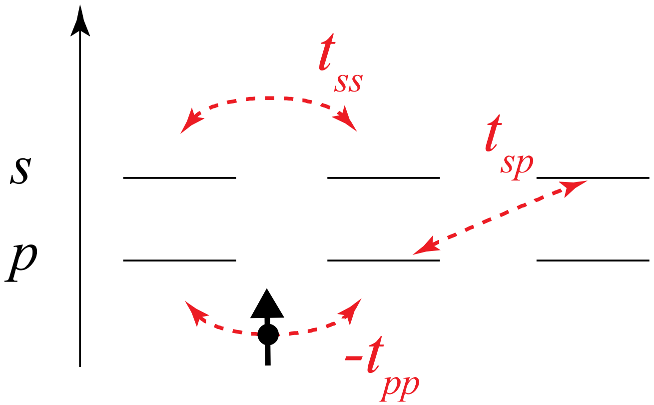




2dTI in HgTe/CdTe quantum well







$$H_{so} = \lambda \, \boldsymbol{L} \cdot \boldsymbol{S}$$

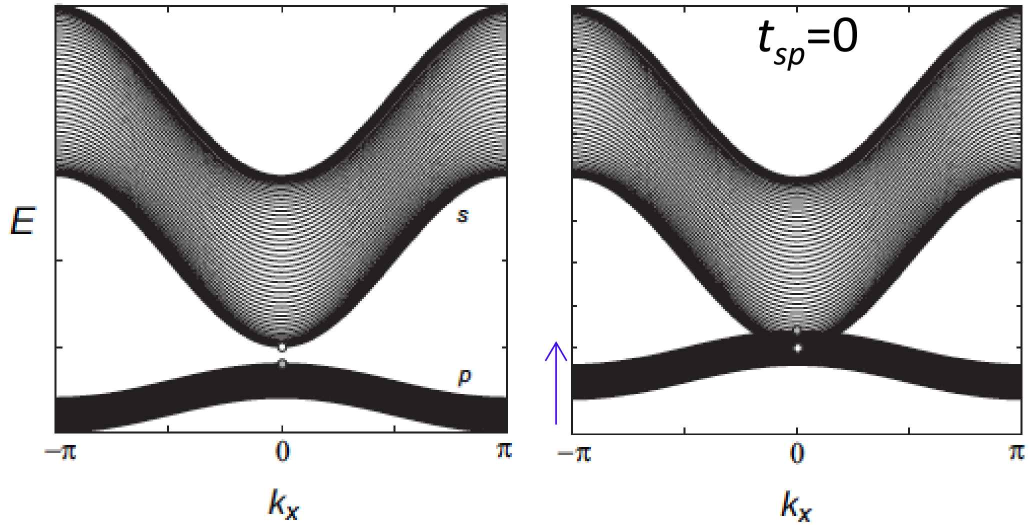
$$= \frac{\lambda}{2}\left(\boldsymbol{J}^2 - \boldsymbol{L}^2 - \boldsymbol{S}^2 \right)$$

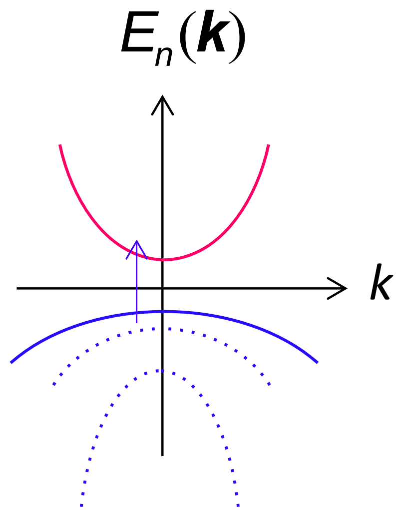
2dTI in HgTe/CdTe quantum well

Weak SOC → Strong SOC

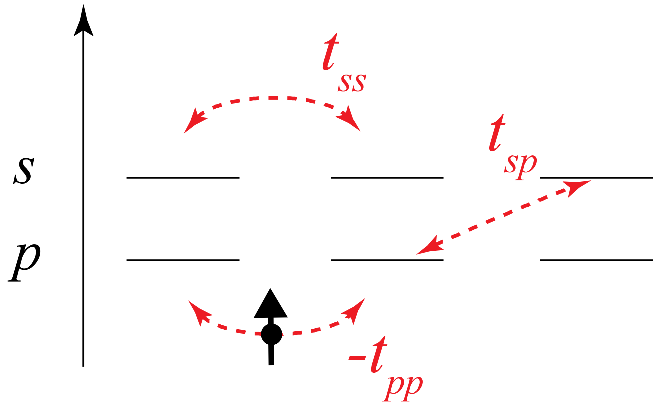
s−band

p−band

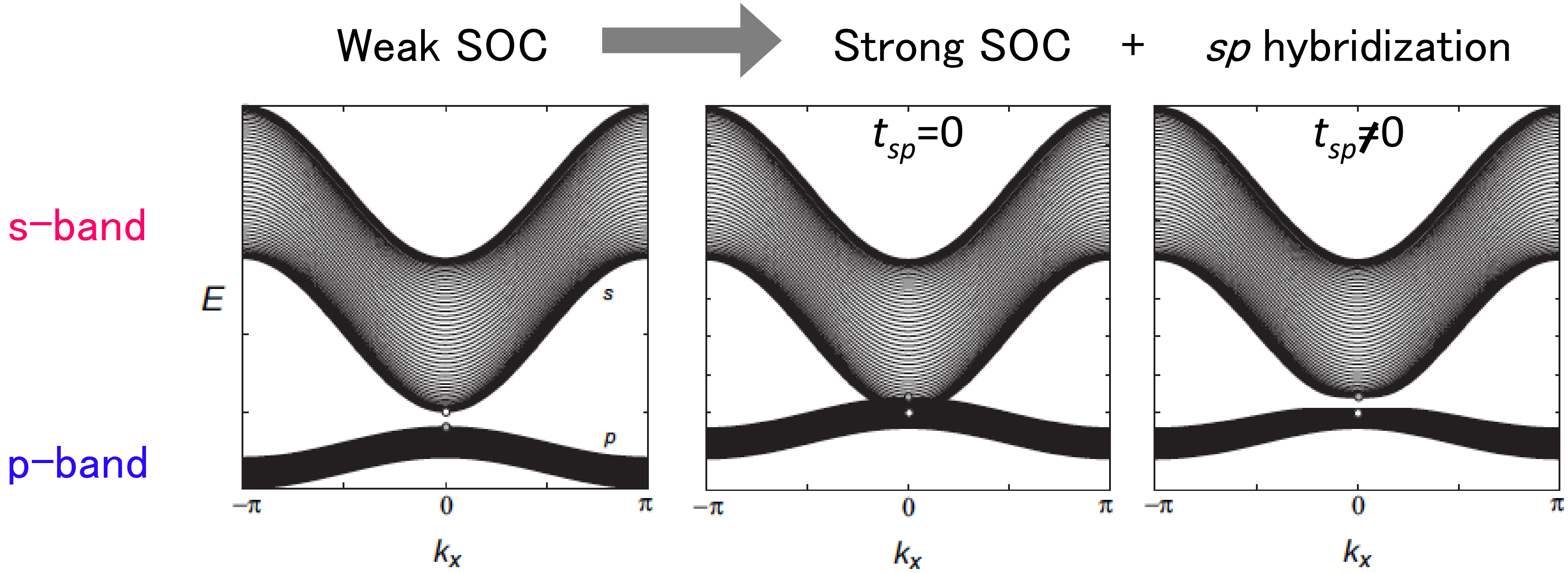




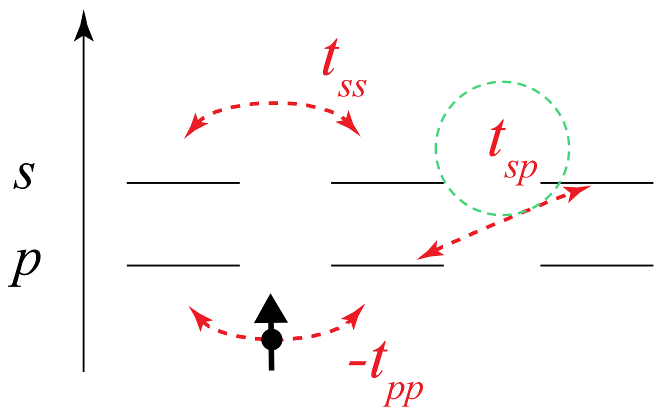
Bernevig, Hughes, Zhang (2006)



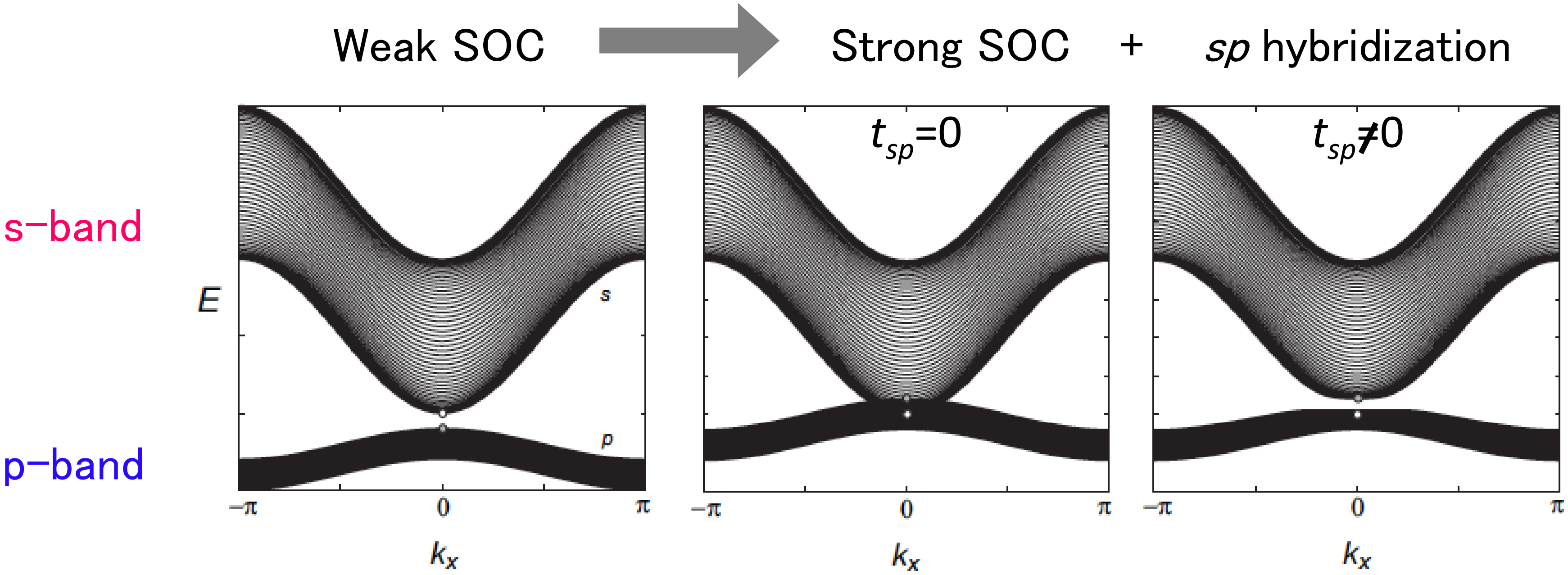
2dTI in HgTe/CdTe quantum well

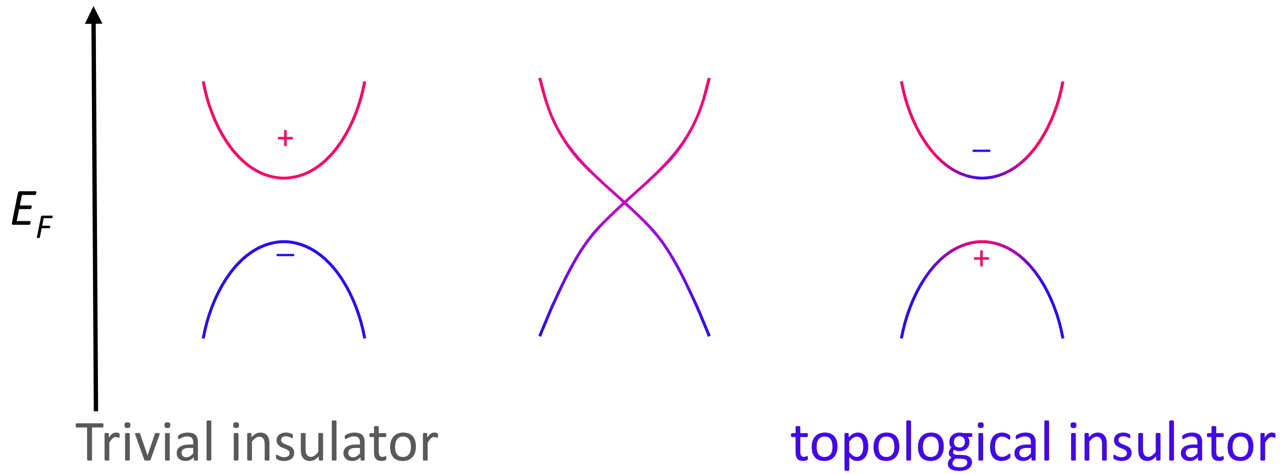


Bernevig, Hughes, Zhang (2006)

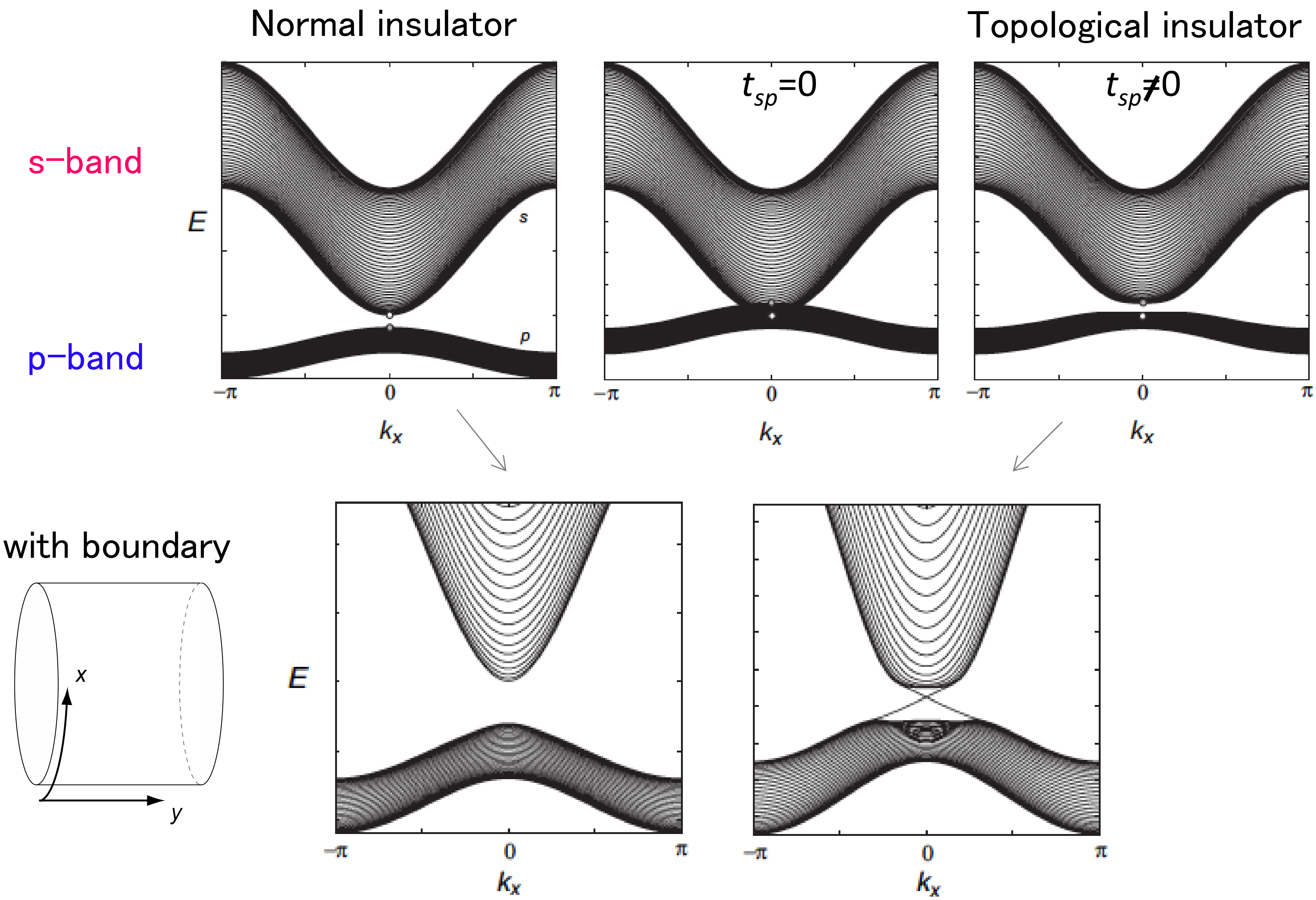


2dTI in HgTe/CdTe quantum well

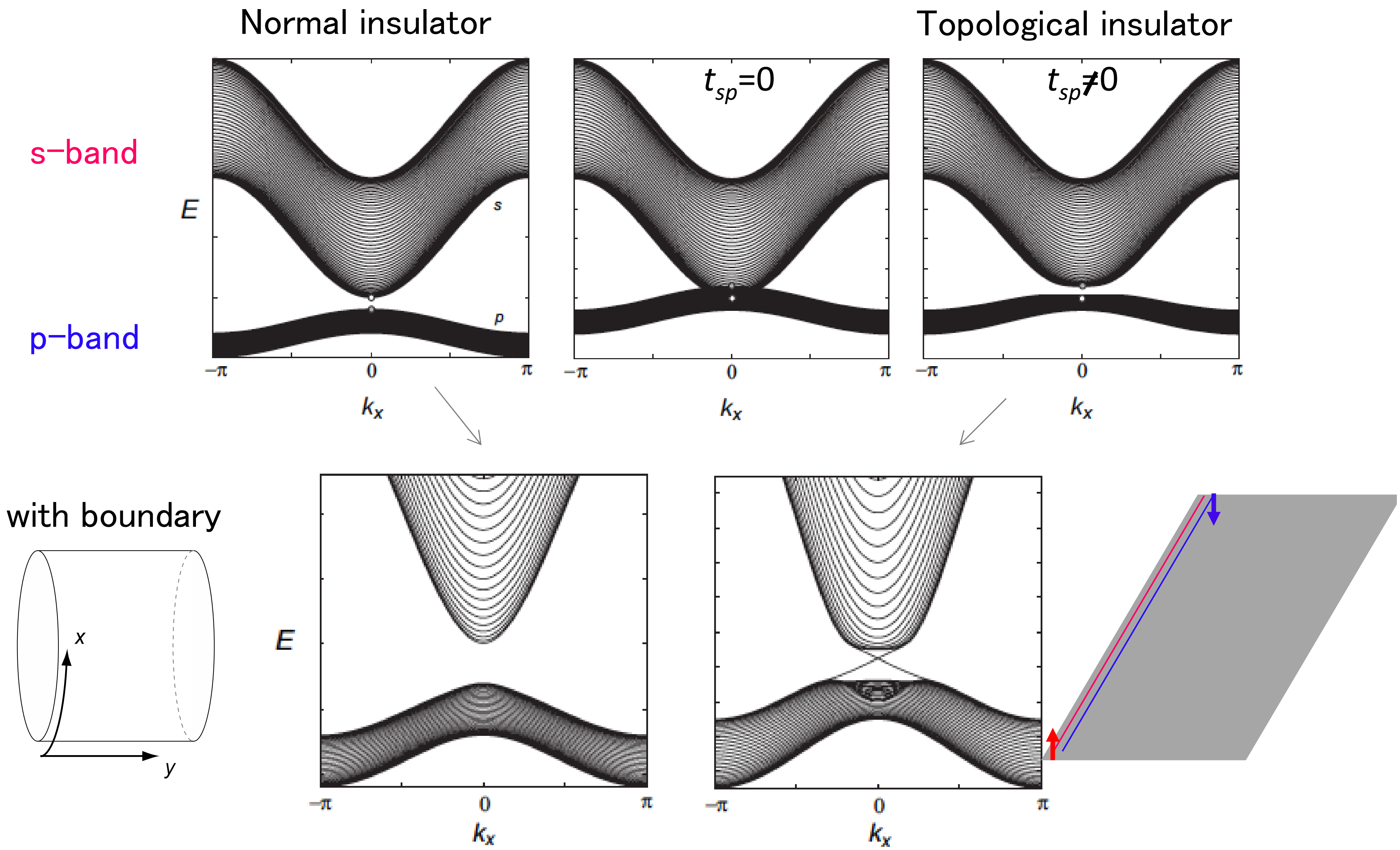




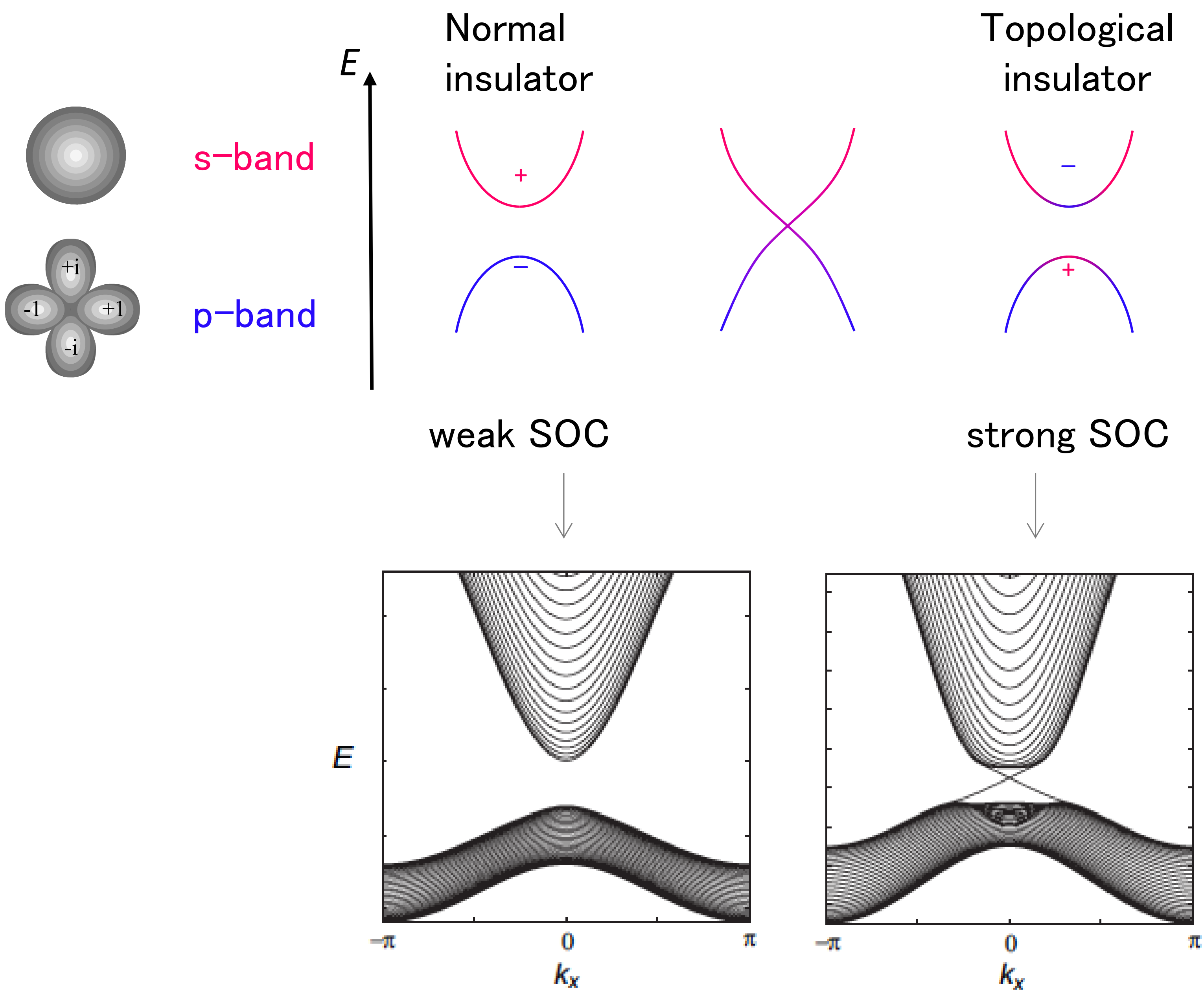
2dTI in HgTe/CdTe quantum well



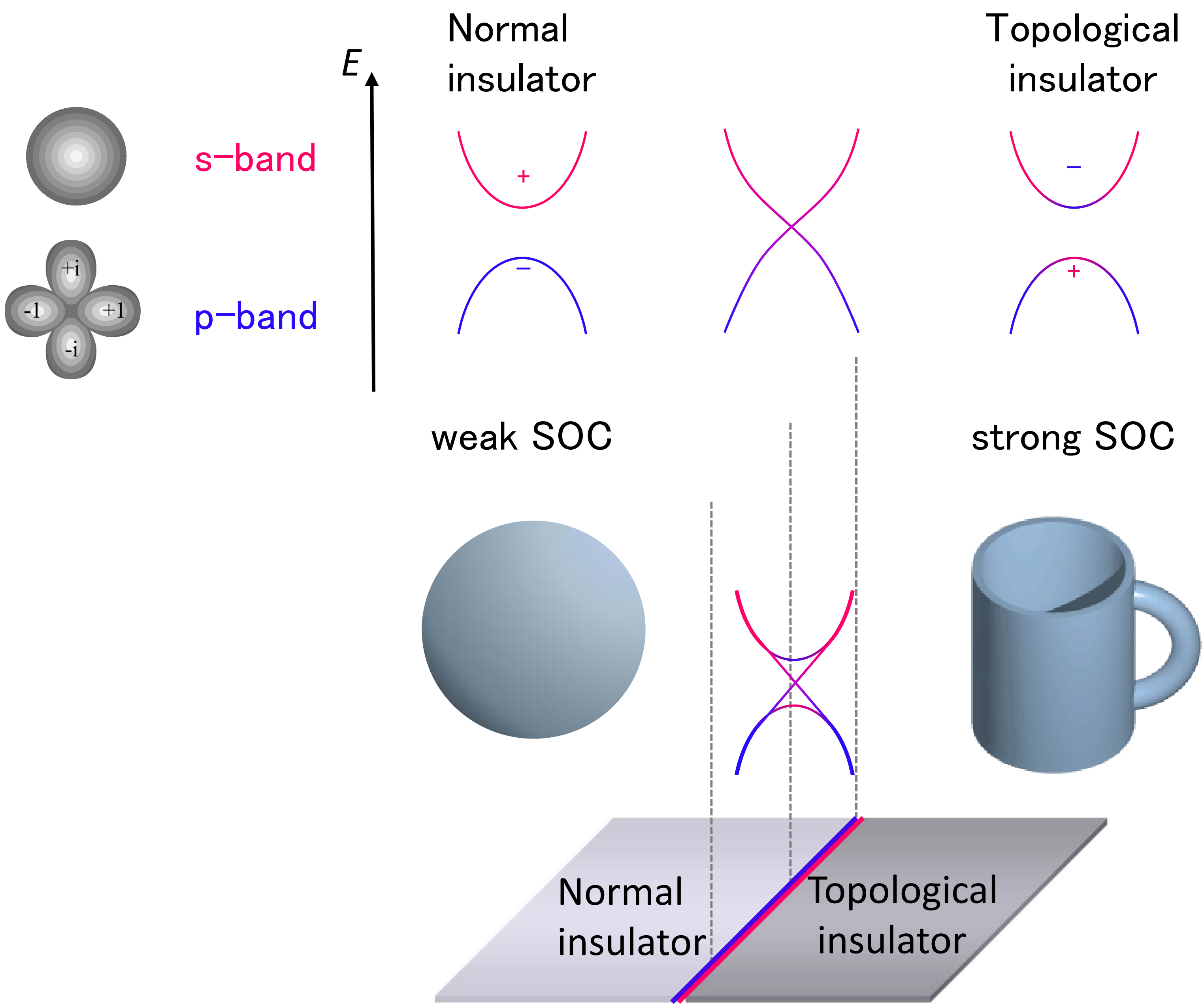
2dTI in HgTe/CdTe quantum well



2dTI in HgTe/CdTe quantum well



2dTI in HgTe/CdTe quantum well

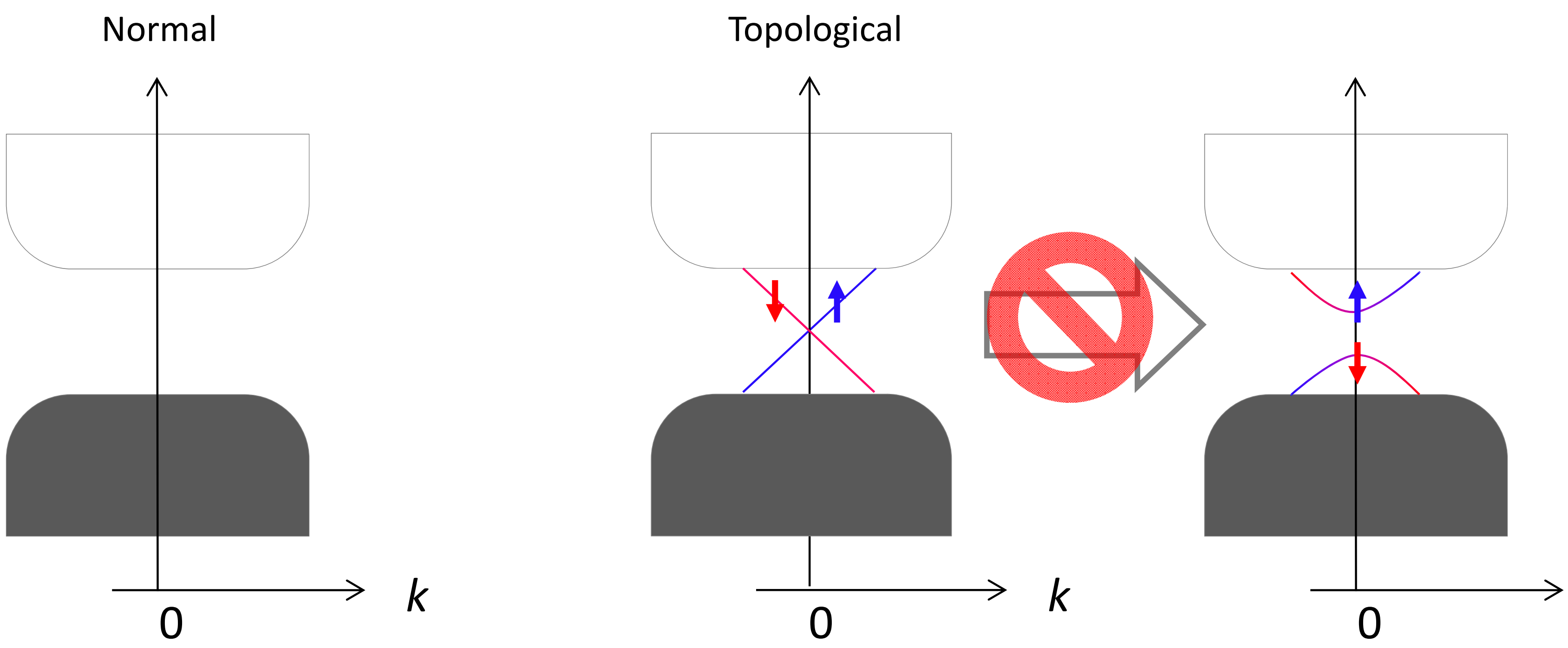


2dTI in HgTe/CdTe quantum well

Time-reversal symmetry

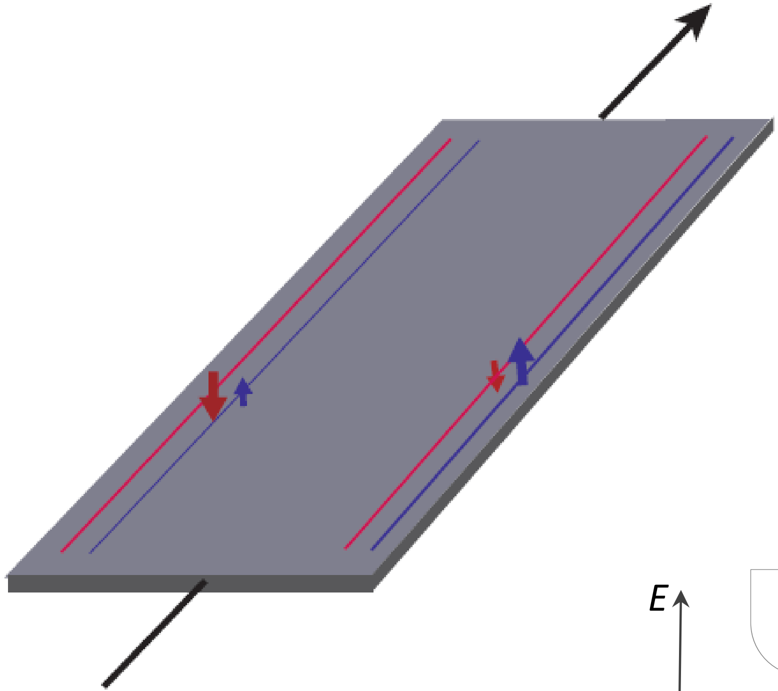
$$E_{k\uparrow} = E_{-k\downarrow}$$

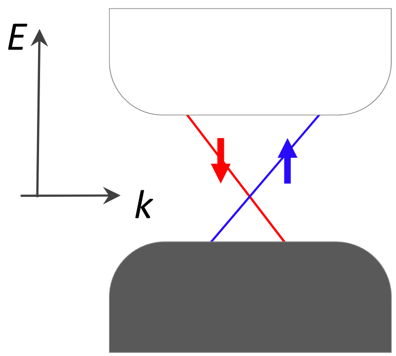
2-fold degeneracy at k=0 is protected by symmetry

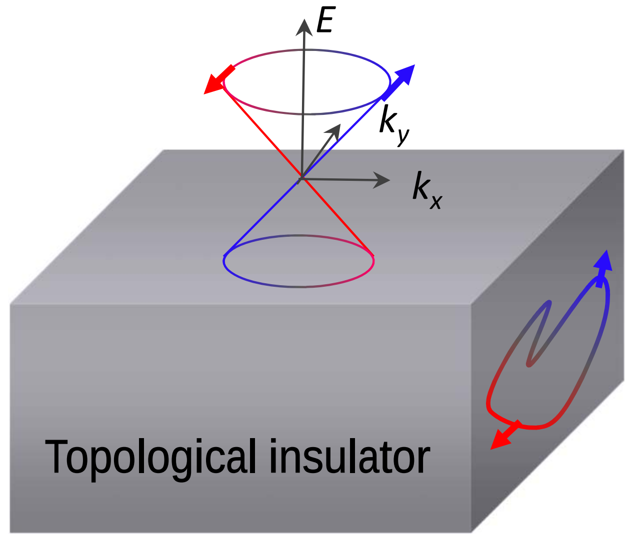


From 2d to 3d

Moore-Balents, Roy, Fu-Kane-Mele, ...







2d

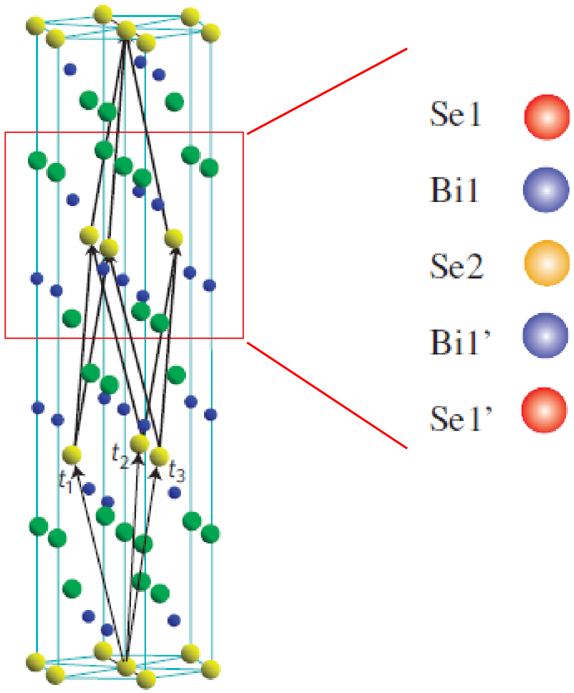
HgTe QW

3d

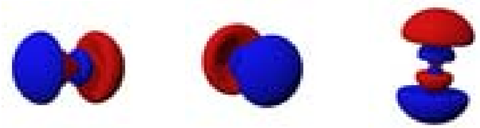
Bi-Sb, Bi_2Se_3, Bi_2Te_3, ...

3d Topological insulator Bi_2Se_3

Zhang et al. '09



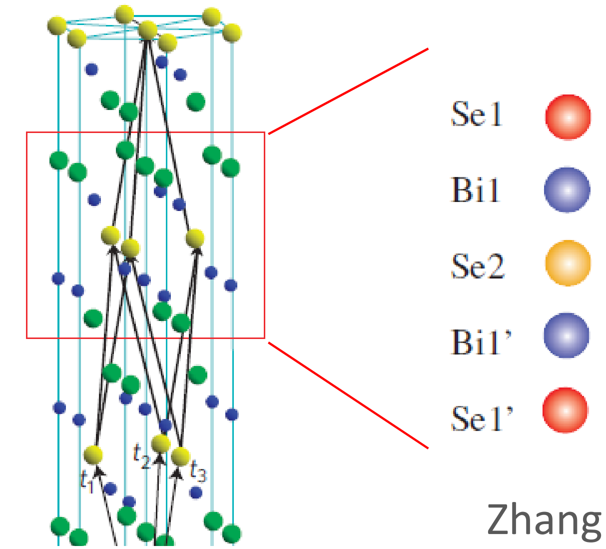
Bi : $6s^2 6p^3$
Se : $4s^2 4p^2$



5 x 3 (p_x,p_y,p_z) x 2 (spin) = 30 *p*-states

						2 He
	5 B	6 C	7 N	8 O	9 F	10 Ne
	13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
112 Cn	113 Uut	114 Fl	115 Uup	116 Lv	117 Uus	118 Uuo

3d Topological insulator Bi_2Se_3



Bi : $6s^2 6p^3$

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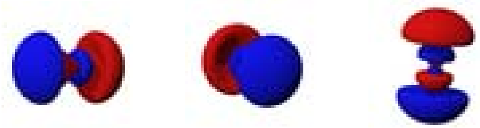
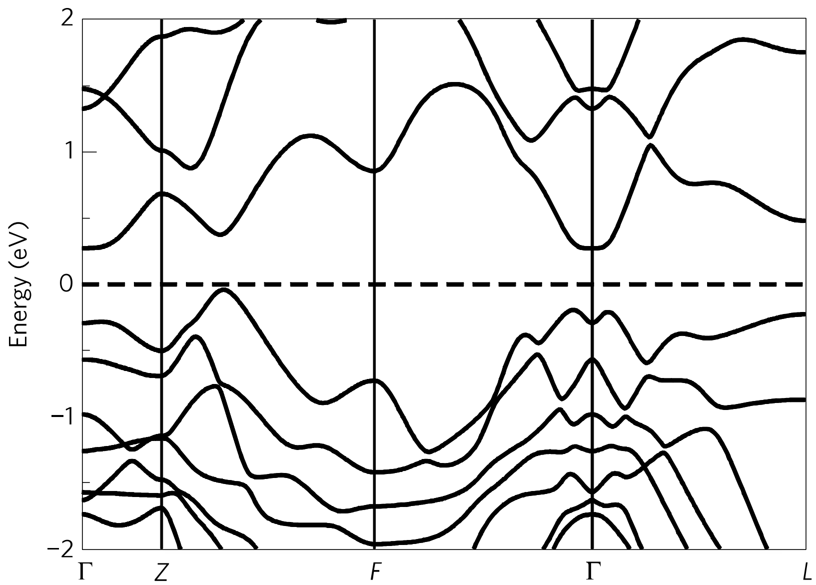
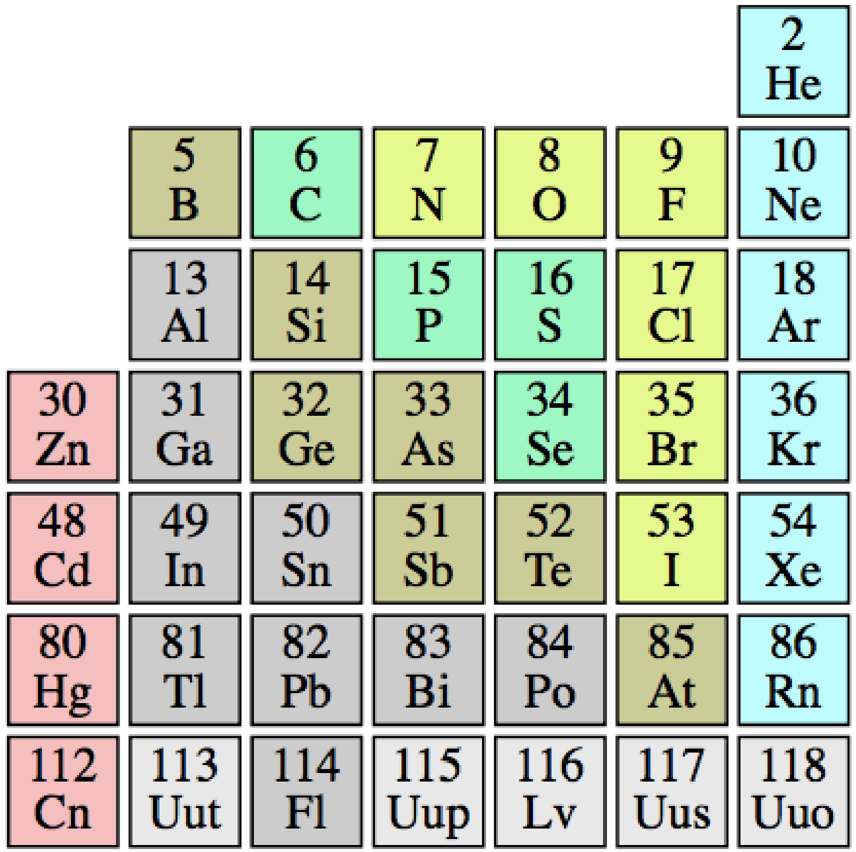


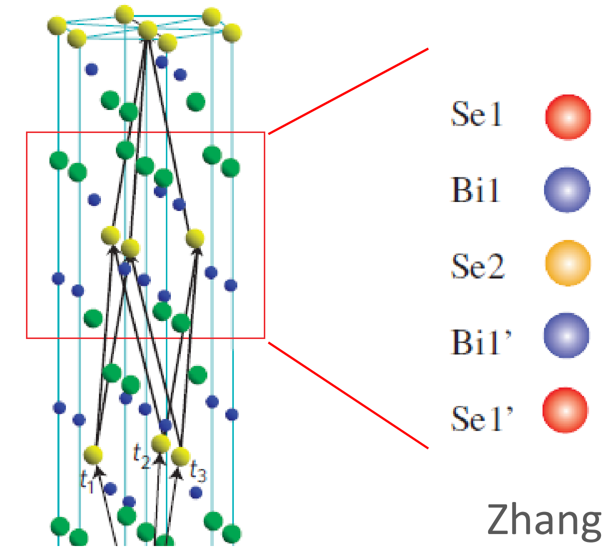
$5 \times 3\ (p_x, p_y, p_z) \times 2$ (spin) = 30 *p*-states

Zhang et al. '09





3d Topological insulator Bi_2Se_3



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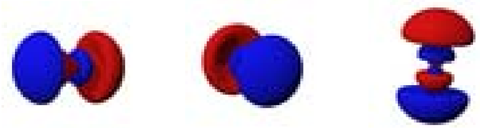
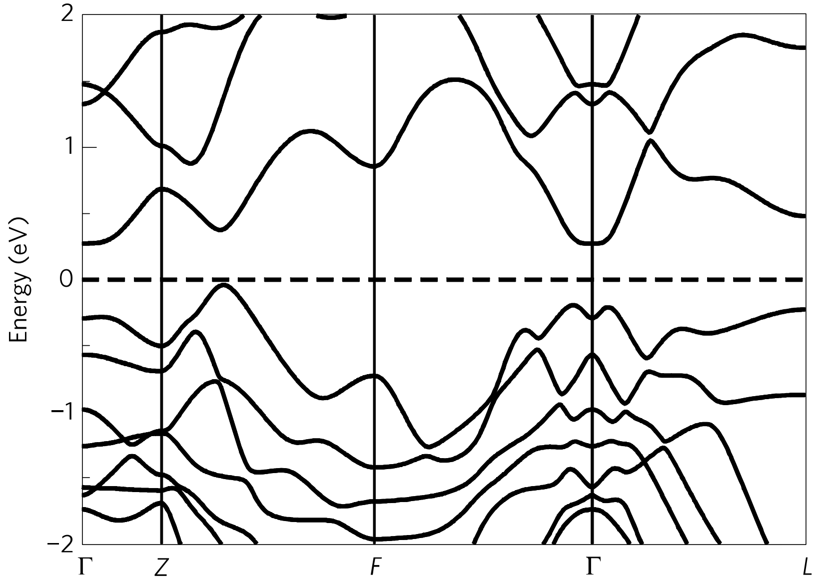


$5 \times 3\ (p_x, p_y, p_z) \times 2\ \text{(spin)} = 30\ p\text{-states}$

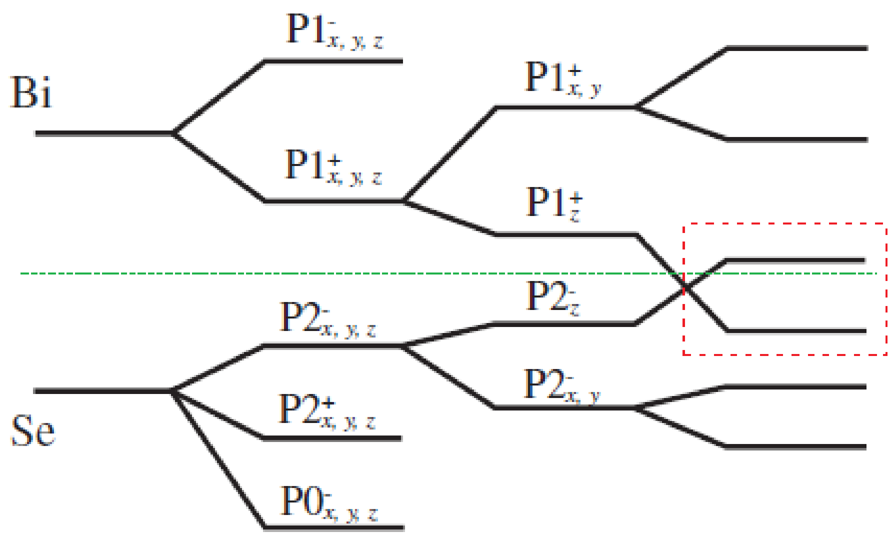
Zhang et al. '09



hybrid orbitals

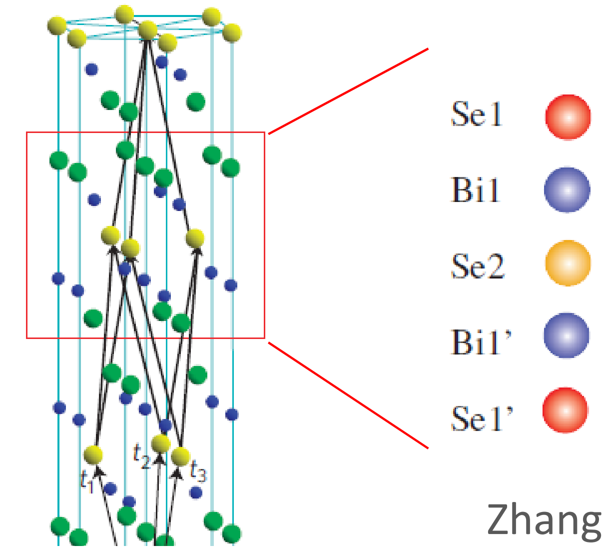
crystal field splitting

SO splitting



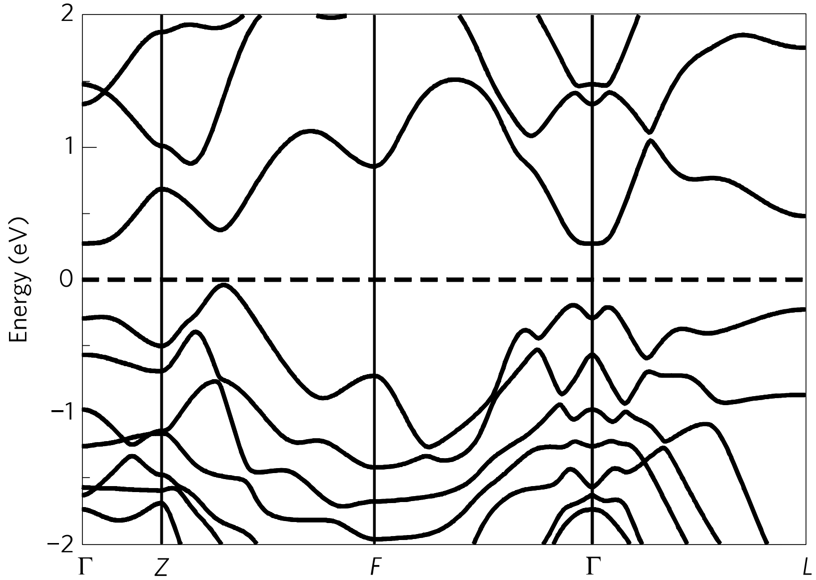
4-band model

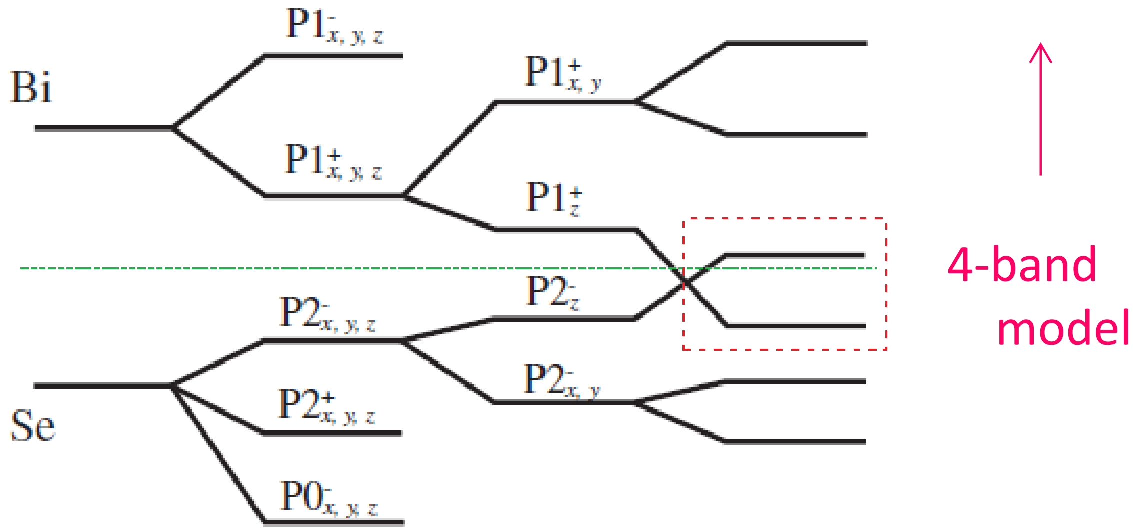
3d Topological insulator Bi_2Se_3



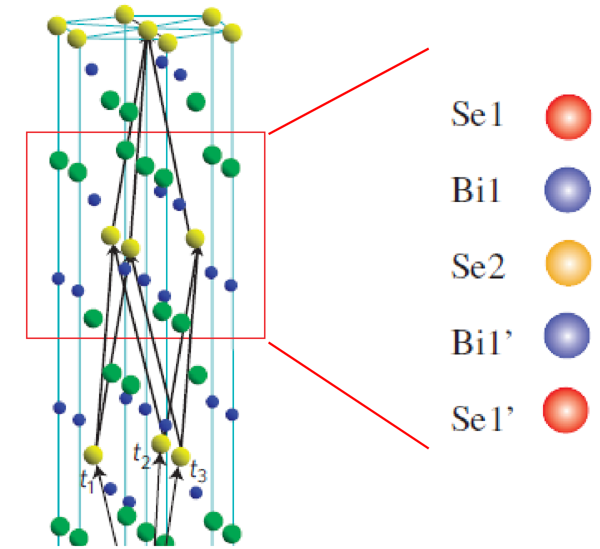
Zhang et al. '09

$$H(\boldsymbol{k}=0) = \begin{pmatrix} m_0 & 0 & 0 & 0 \\ 0 & m_0 & 0 & 0 \\ 0 & 0 & -m_0 & 0 \\ 0 & 0 & 0 & -m_0 \end{pmatrix} + \varepsilon(k=0)$$





3d Topological insulator Bi_2Se_3



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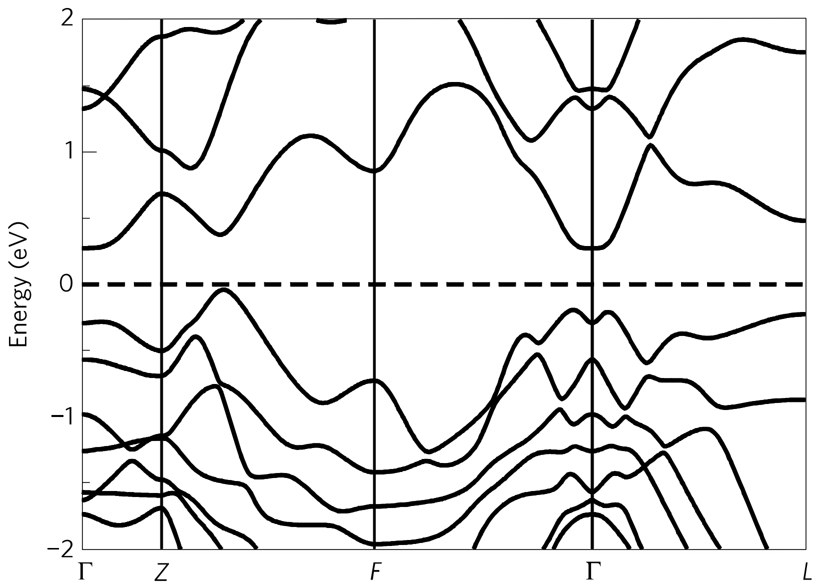
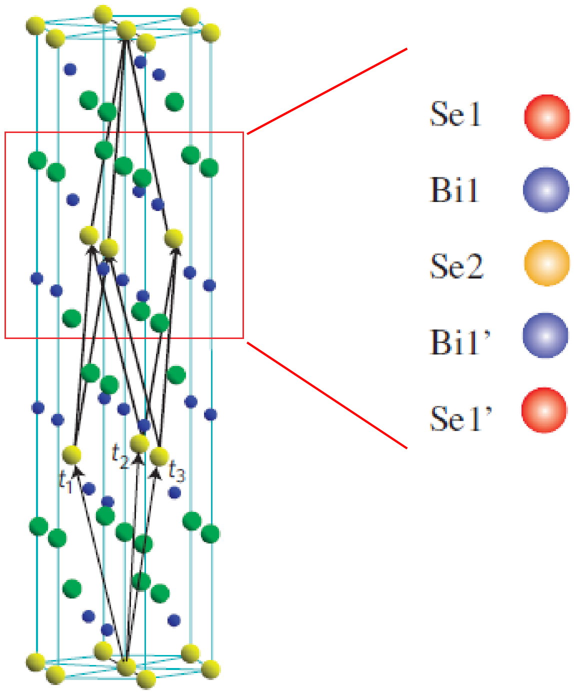


k.p theory

$$H(\boldsymbol{k}) = e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{x})\right]e^{+i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$= \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{x})\right] + \frac{\hbar}{m}\boldsymbol{k}\cdot(-i\hbar\nabla) + \frac{\hbar^2}{2m}\boldsymbol{k}^2$$

3d Topological insulator Bi_2Se_3



$$H(\mathbf{k}) = \begin{pmatrix} m(\mathbf{k}) & 0 & A_1 k_z & A_2 k_- \\ 0 & m(\mathbf{k}) & A_2 k_+ & -A_1 k_z \\ A_1 k_z & A_2 k_+ & -m(\mathbf{k}) & 0 \\ A_2 k_- & -A_1 k_z & 0 & -m(\mathbf{k}) \end{pmatrix} + \varepsilon(k)$$

$$m(\mathbf{k}) = m_0 + \sum_i c_i k_i^2$$

k.p theory

1st order

2nd order

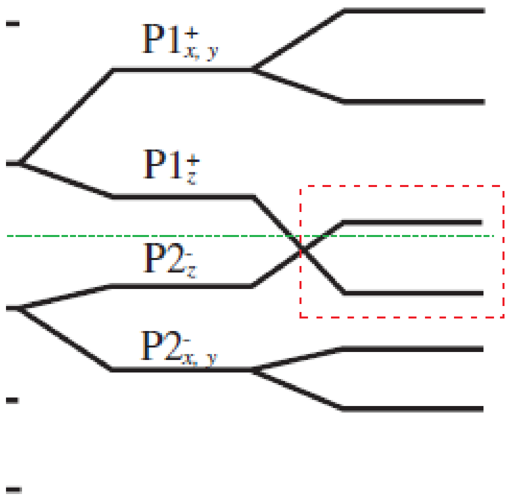
$$H(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{x}} \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) \right] e^{+i\mathbf{k}\cdot\mathbf{x}}$$

$$= \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) \right] + \frac{\hbar}{m} \mathbf{k} \cdot (-i\hbar\nabla) + \frac{\hbar^2}{2m} \mathbf{k}^2$$

3d Topological insulator Bi_2Se_3

$$H(\boldsymbol{k}) = \begin{pmatrix} m(\boldsymbol{k}) & 0 & A_1 k_z & A_2 k_- \\ 0 & m(\boldsymbol{k}) & A_2 k_+ & -A_1 k_z \\ A_1 k_z & A_2 k_+ & -m(\boldsymbol{k}) & 0 \\ A_2 k_- & -A_1 k_z & 0 & -m(\boldsymbol{k}) \end{pmatrix} + \varepsilon(k)$$

$$m(\boldsymbol{k}) = m_0 + \sum_i c_i k_i^2$$



normal insulator

$m_0 > 0$

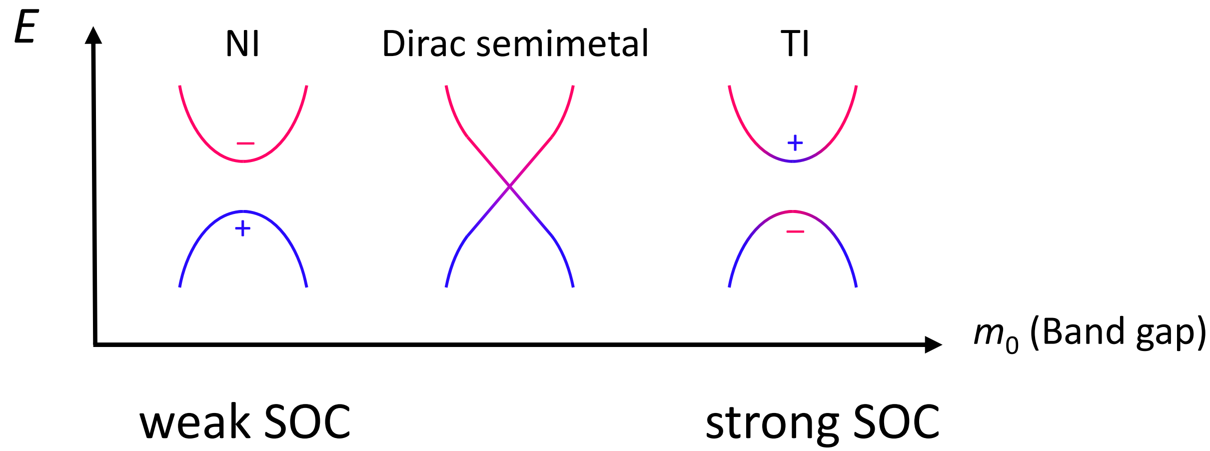
topological insulator

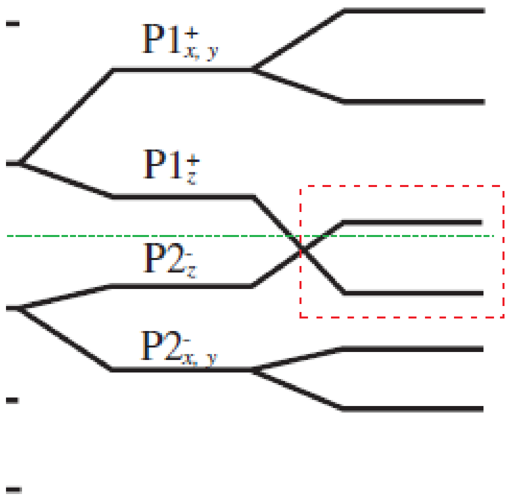
$m_0 < 0$

$m_0 = 0$

The gap vanishes at this point

3d Topological insulator Bi_2Se_3





normal insulator

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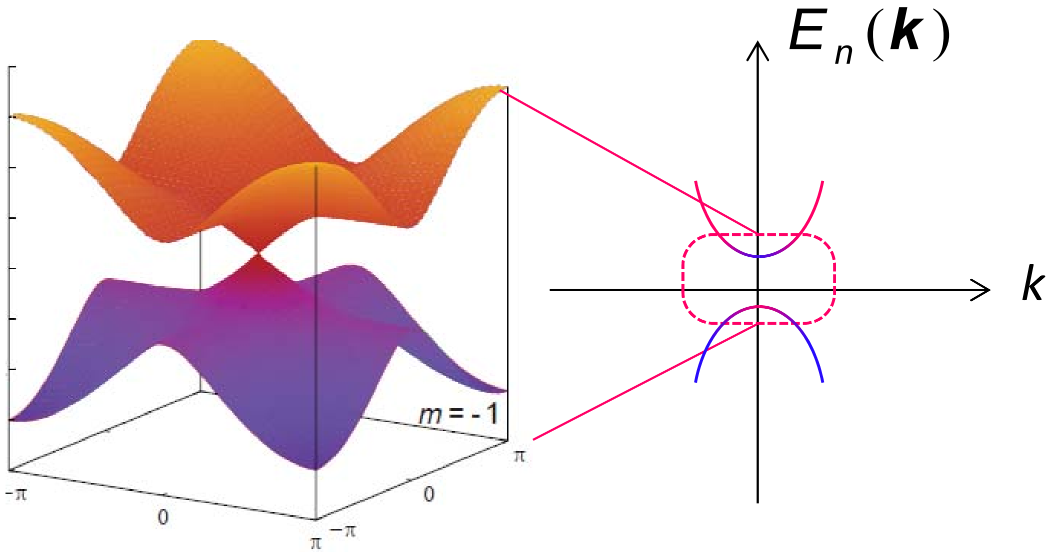
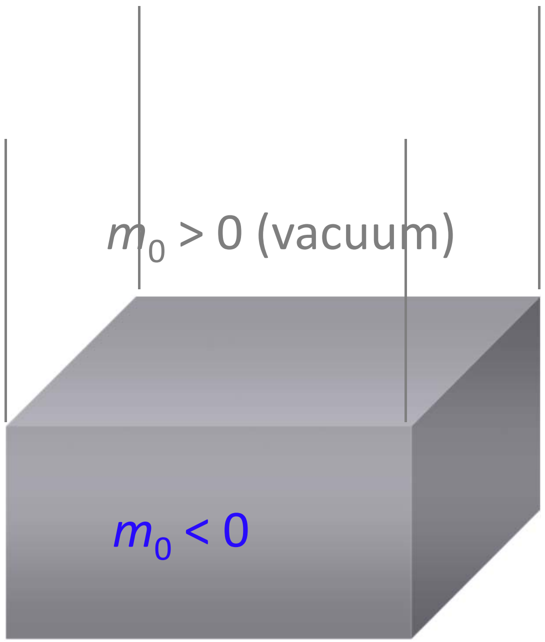
3d Topological insulator Bi_2Se_3

Surface Dirac modes
realized in a slab geometry





normal insulator

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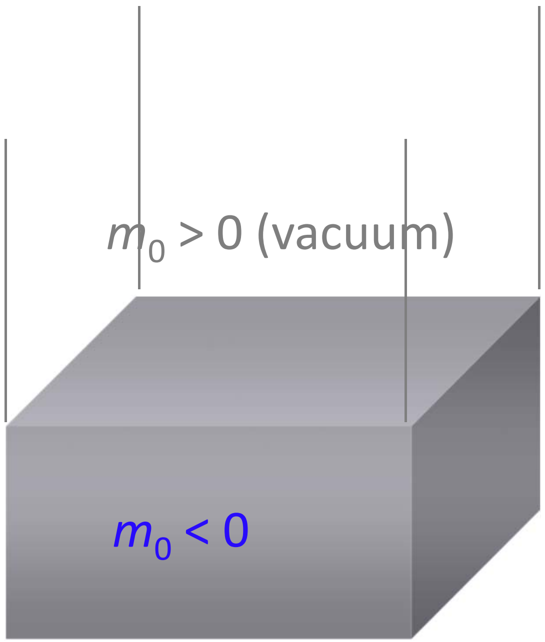
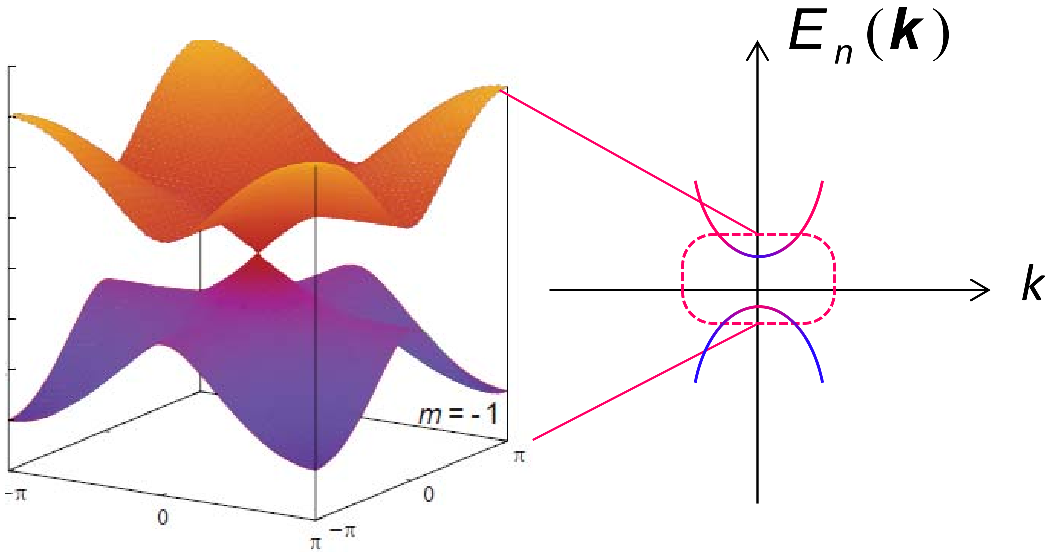
3d Topological insulator Bi_2Se_3

Surface Dirac modes realized in a slab geometry





Surface modes described by the Dirac Hamiltonian:

$$H = i\hbar v \sigma_y \frac{\partial}{\partial x} - i\hbar v \sigma_x \frac{\partial}{\partial y} + m\sigma_z$$

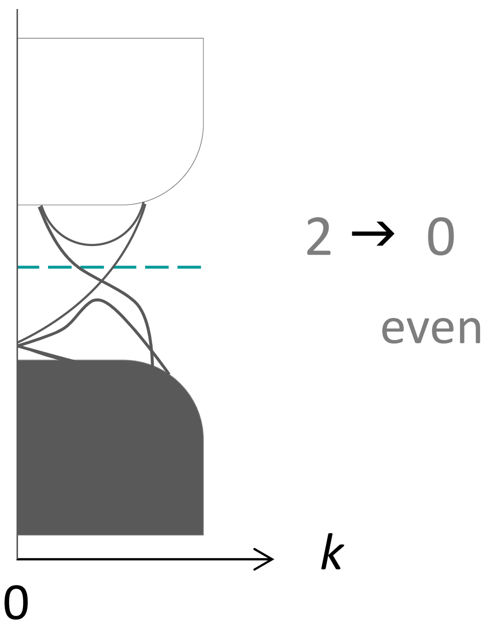
$(m = 0)$

Z_2 topological insulators

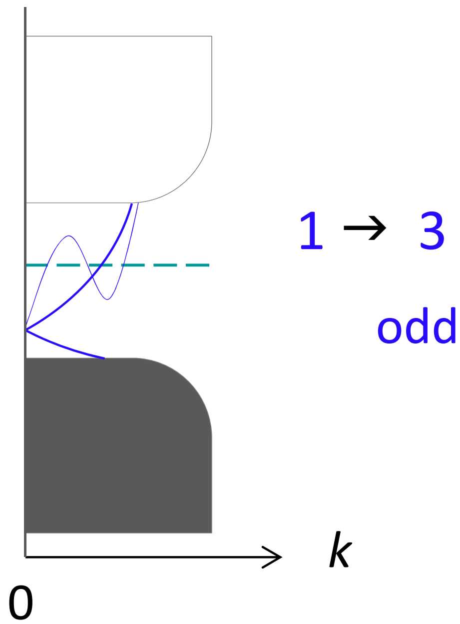
Z_2 = { 0, 1 }

even or odd

weak topological insulator
(ordinary insulator)



strong topological insulator

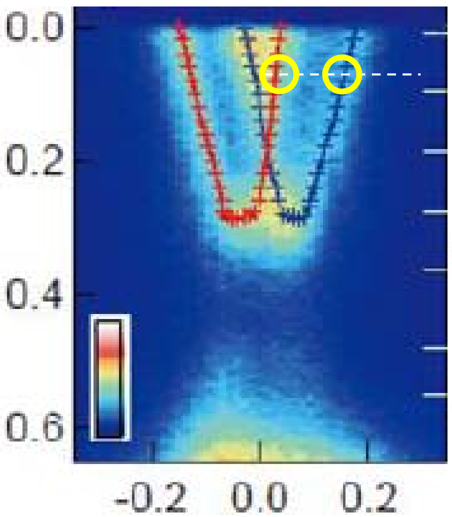


Z_2 topological insulators

$$Z_2 = \{\, 0, 1 \,\}$$

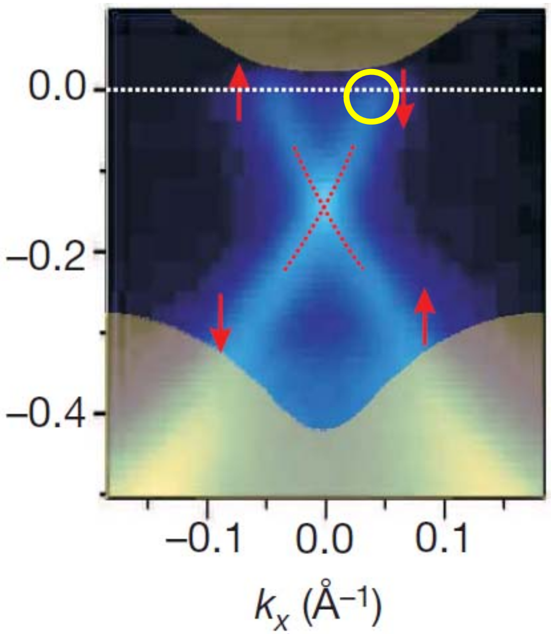
even or odd

Ishizaka et al. (2011)



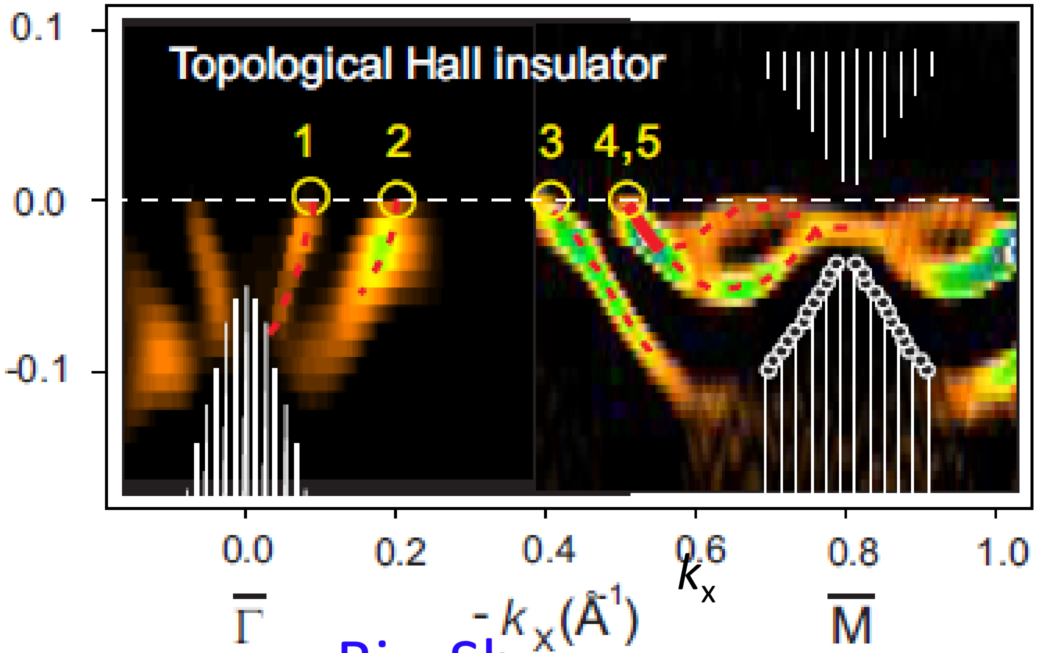
BiTeI

Hsieh et al. (2009)



Bi_2Se_3

Hsieh et al. (2008)

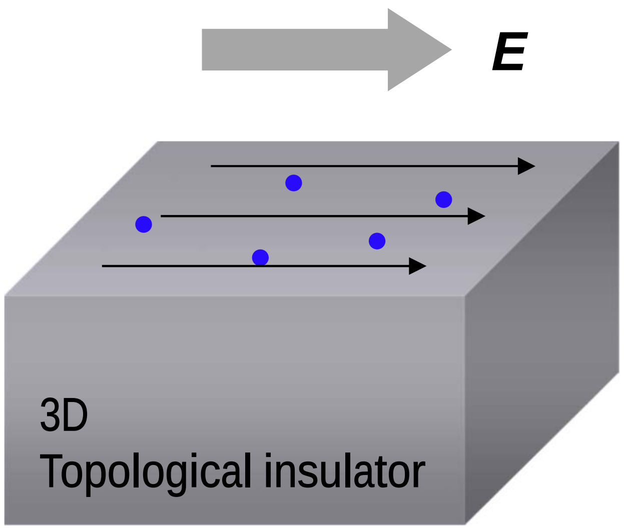


$Bi_{1-x}Sb_x$

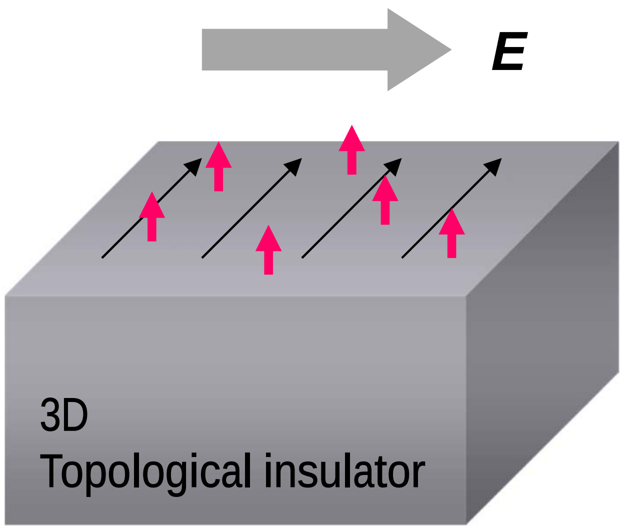
Impurity effects

$$H_{surface} = v_F(\sigma_y p_x - \sigma_x p_y) + V_0(\boldsymbol{r}) + \boldsymbol{\sigma}\cdot\boldsymbol{V}(\boldsymbol{r})$$

non-magnetic impurities



magnetic impurities

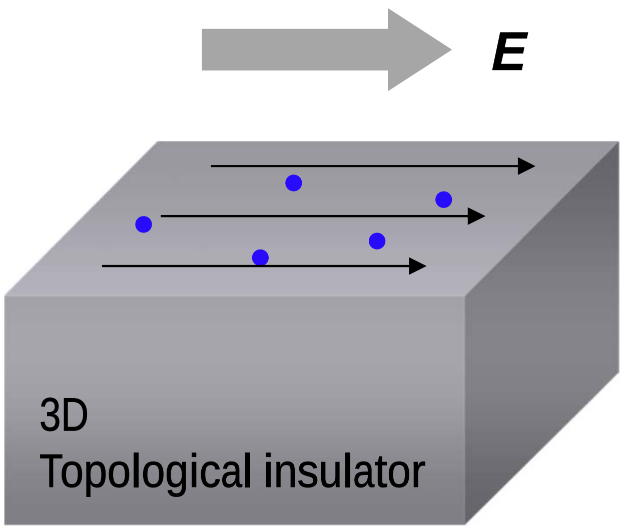


Impurity effects

$$H_{surface} = v_F(\sigma_y p_x - \sigma_x p_y) + V_0(\mathbf{r}) + \sigma \cdot \mathbf{V}(\mathbf{r})$$

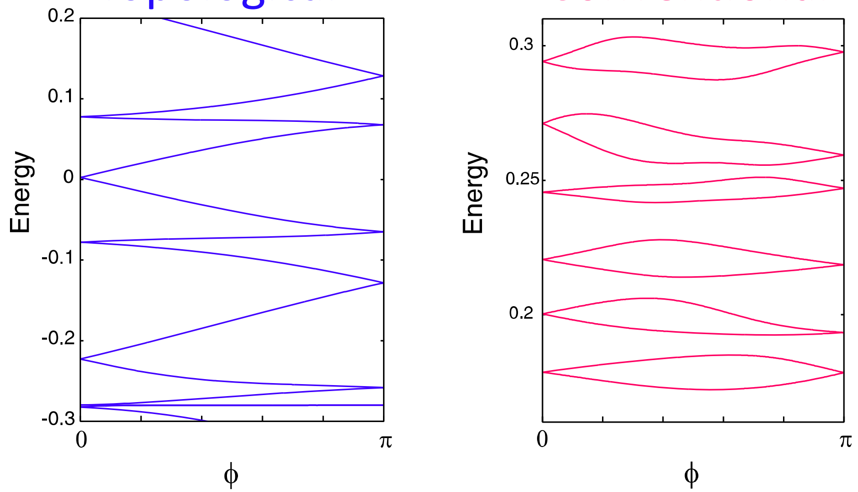
non-magnetic impurities

KN, Koshino, Ryu, PRL (2007)



Topologically protected from Anderson localization

Topological   Conventional



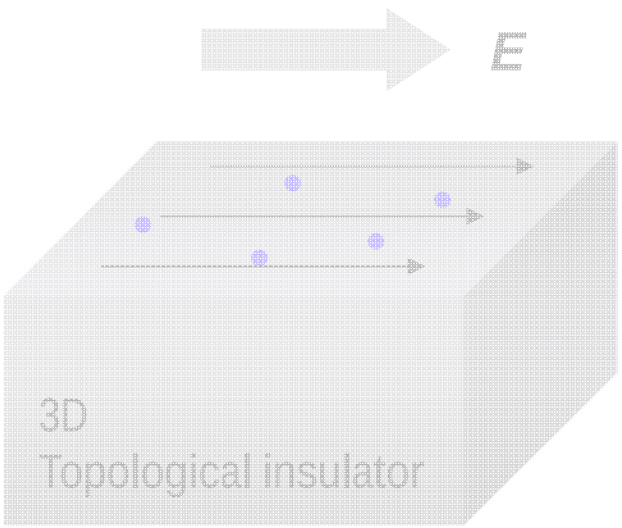
Z_2 odd Z_2 even

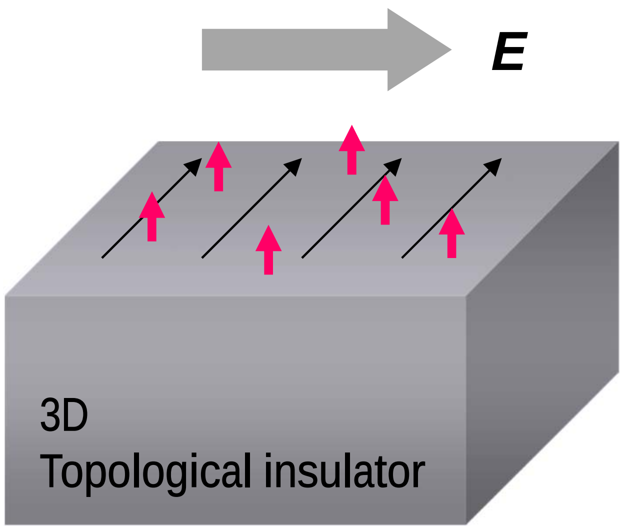
Impurity effects

$$H_{surface} = v_F(\sigma_y p_x - \sigma_x p_y) + V_0(\boldsymbol{r}) + \boldsymbol{\sigma}\cdot\boldsymbol{V}(\boldsymbol{r})$$

non-magnetic impurities

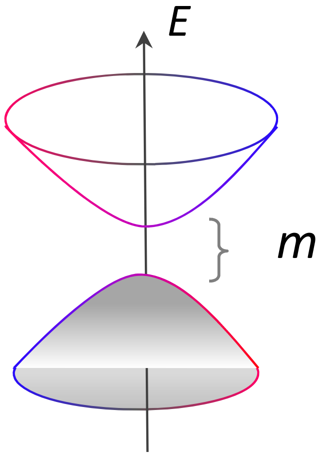
magnetic impurities





Ideal uniform case

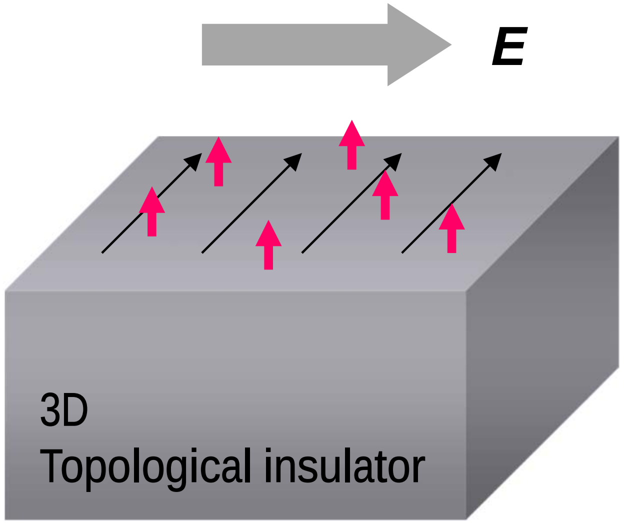
$$H_{surface} = v_F(\sigma_y p_x - \sigma_x p_y) + V_0(\boldsymbol{r}) + \boldsymbol{\sigma}\cdot\boldsymbol{V}(\boldsymbol{r})$$



$$b_z(\boldsymbol{k}) = \frac{m}{2\sqrt{k^2+m^2}^3}$$

$$\frac{\sigma_{xy}}{e^2/h} = \int \frac{d^2k}{2\pi}\, b_z(\boldsymbol{k}) = \frac{1}{2}$$

magnetic impurities

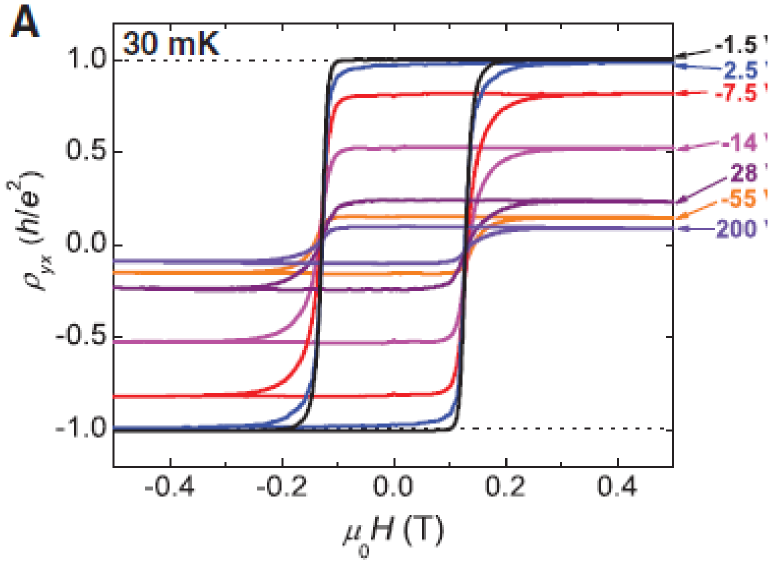


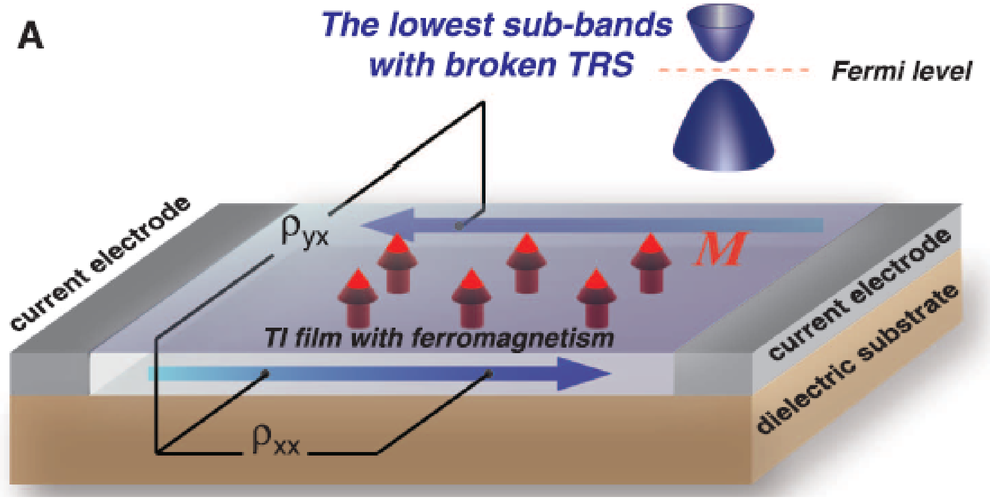
Quantum Anomalous Hall Effect

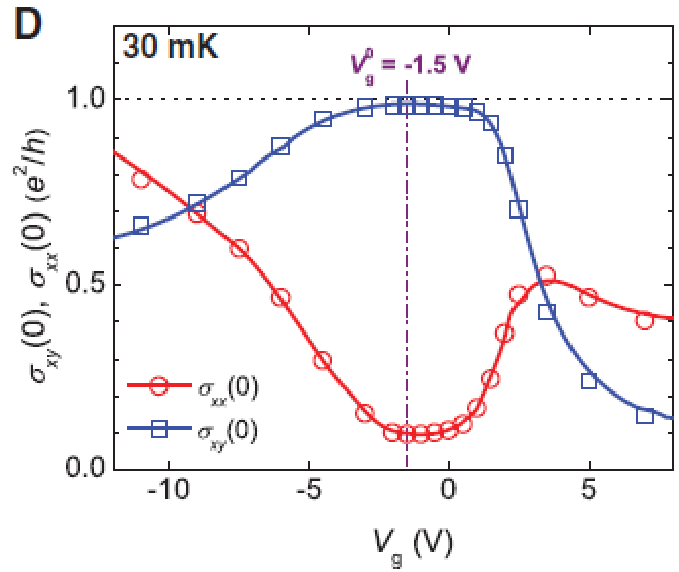
Science **340**, 167 (2013)

Experimental Observation of the Quantum Anomalous Hall Effect in a Magnetic Topological Insulator

Cui-Zu Chang,[1,2*] Jinsong Zhang,[1*] Xiao Feng,[1,2*] Jie Shen,[2*] Zuocheng Zhang,[1] Minghua Guo,
Kang Li,[2] Yunbo Ou,[2] Pang Wei,[2] Li-Li Wang,[2] Zhong-Qing Ji,[2] Yang Feng,[1] Shuaihua Ji,[1]
Xi Chen,[1] Jinfeng Jia,[1] Xi Dai,[2] Zhong Fang,[2] Shou-Cheng Zhang,[3] Ke He,[2]† Yayu Wang,[1]† Li Lu
Xu-Cun Ma,[2] Qi-Kun Xue[1]†



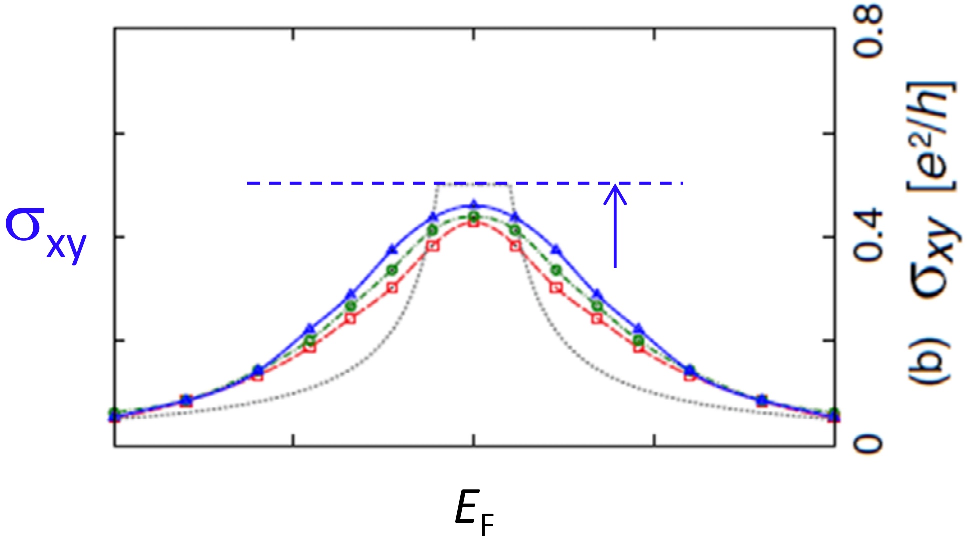


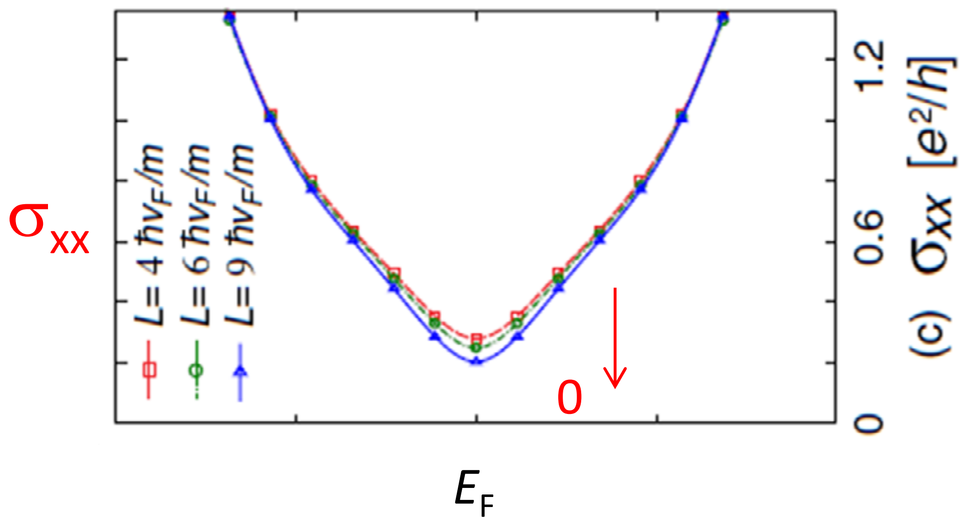


Quantum Anomalous Hall Effect

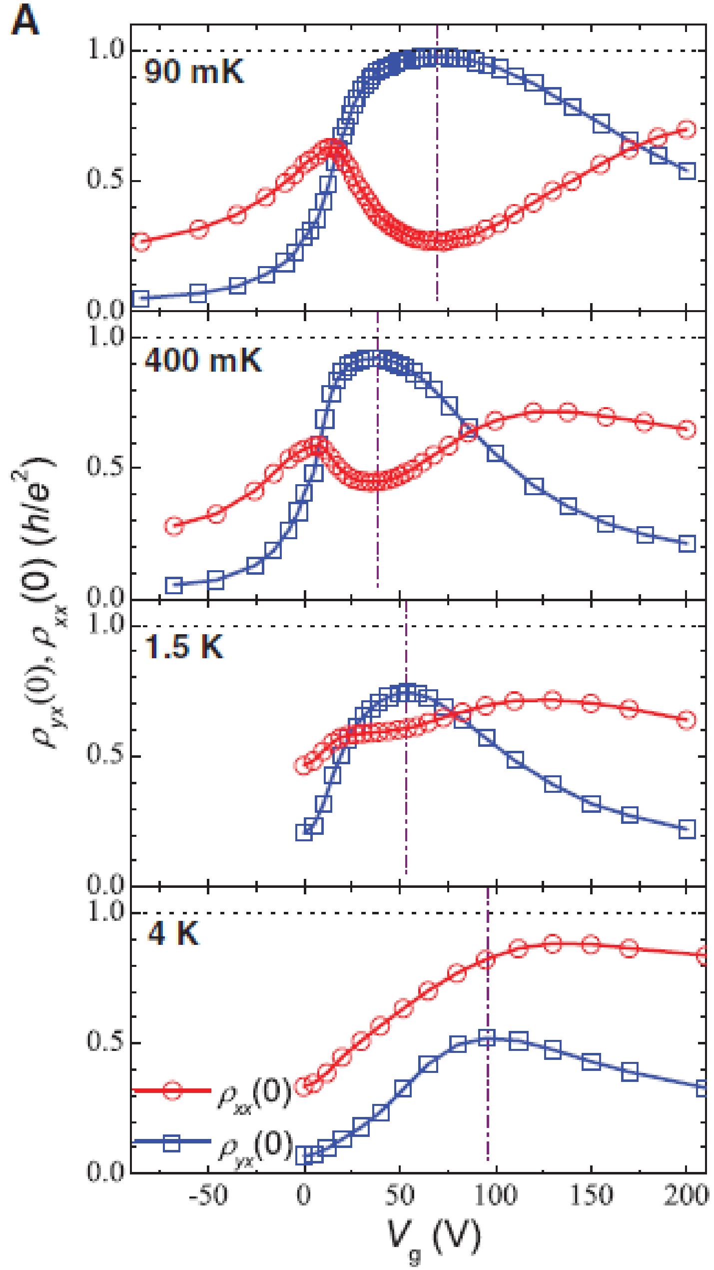
Theory

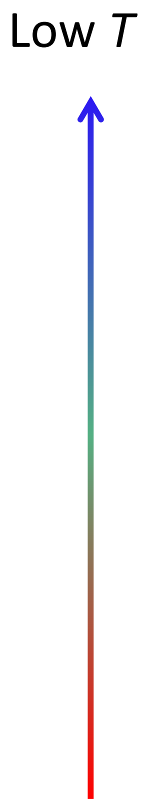
KN, Nagaosa (2011)





Experiment (2013)





Quantum Anomalous Hall Effect

Theory

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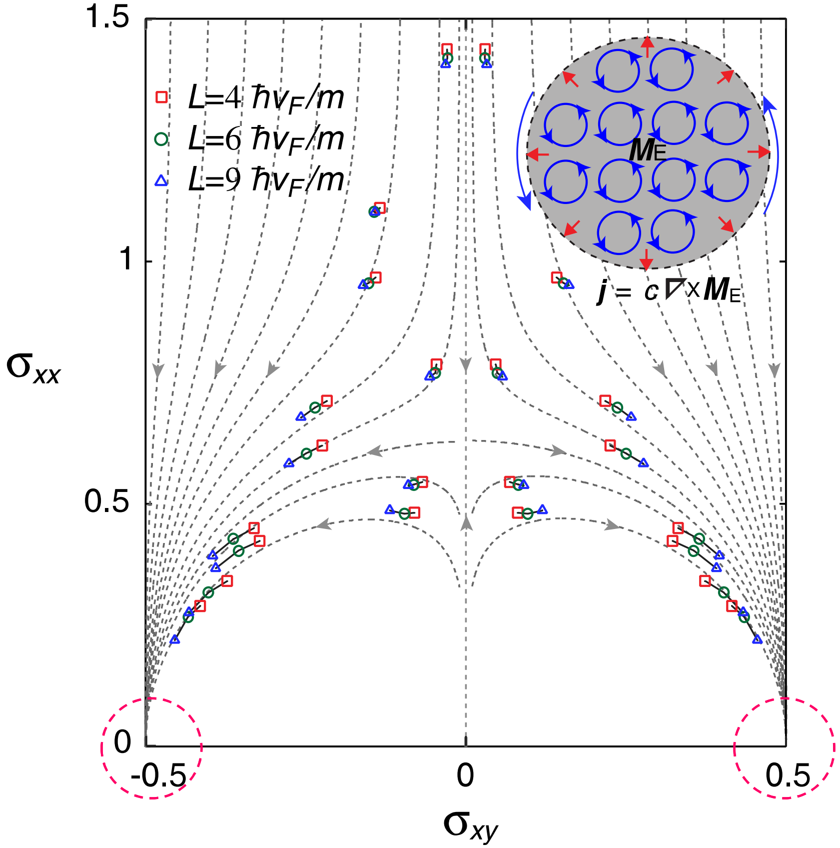
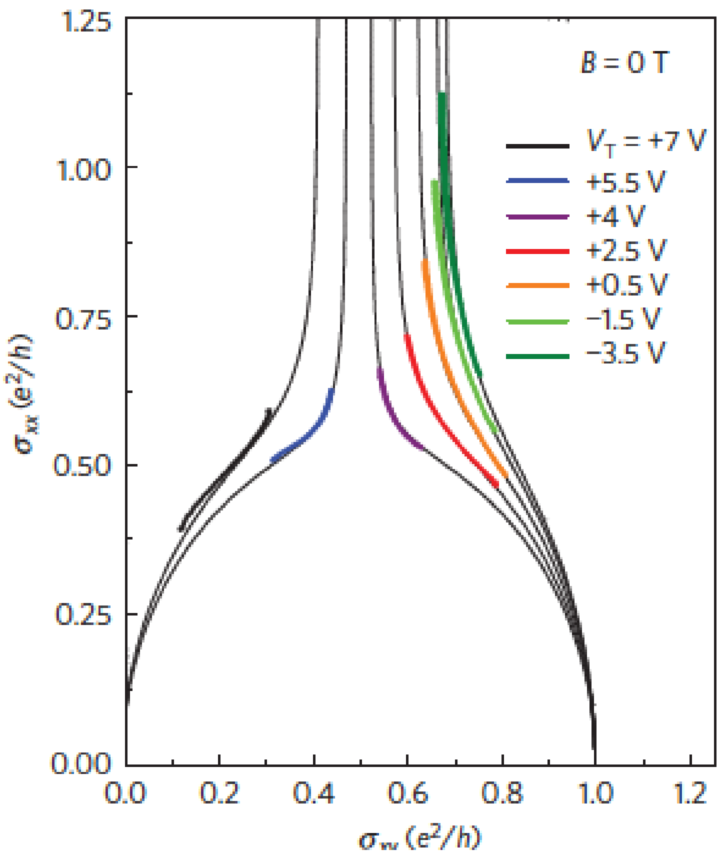


Experiment

Checkelsky, Yoshimi, Tsukazaki, et al. (2014)



Lecture at JAEA 1/23/2017 RIKEN

Introduction to Topological Insulators

Kentaro Nomura (IMR, Tohoku)

outline

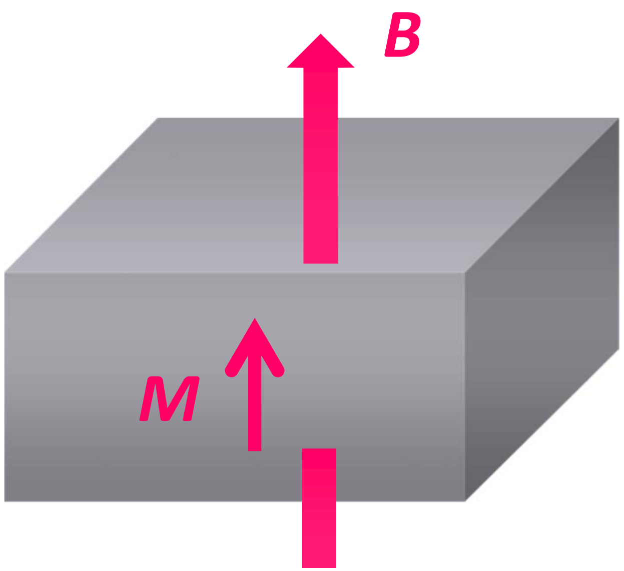
- Quantum Hall effect
- Z2 topological insulators
- Electromagnetic responses

Response to ElectroMagnetic fields

in normal insulators

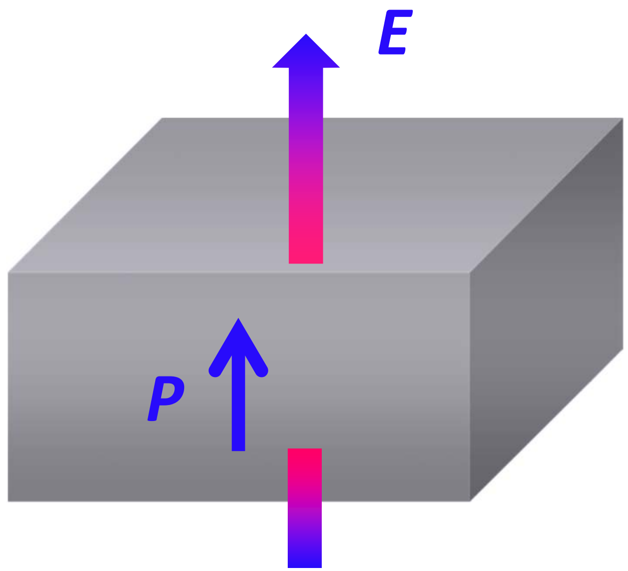
$$\boldsymbol{M} = \chi_m \boldsymbol{B}$$

(magnetization)



$$\boldsymbol{P} = \chi_e \boldsymbol{E}$$

(Electric polarization)

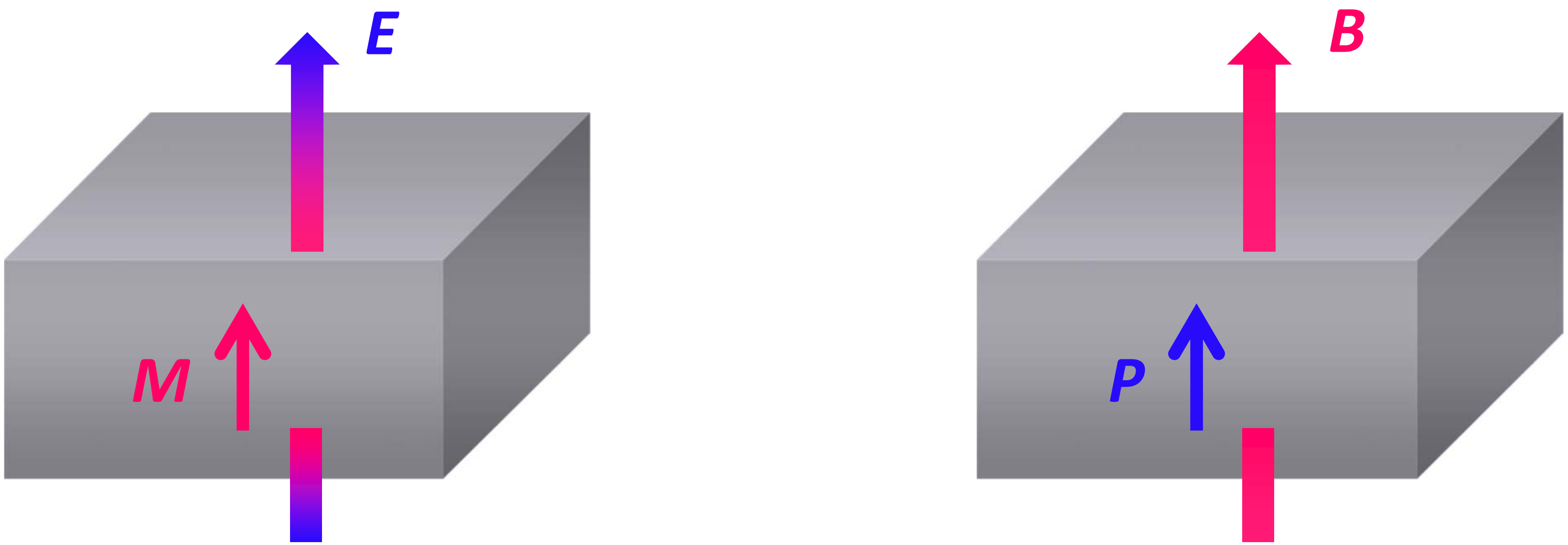


Response to ElectroMagnetic fields

in topological insulators

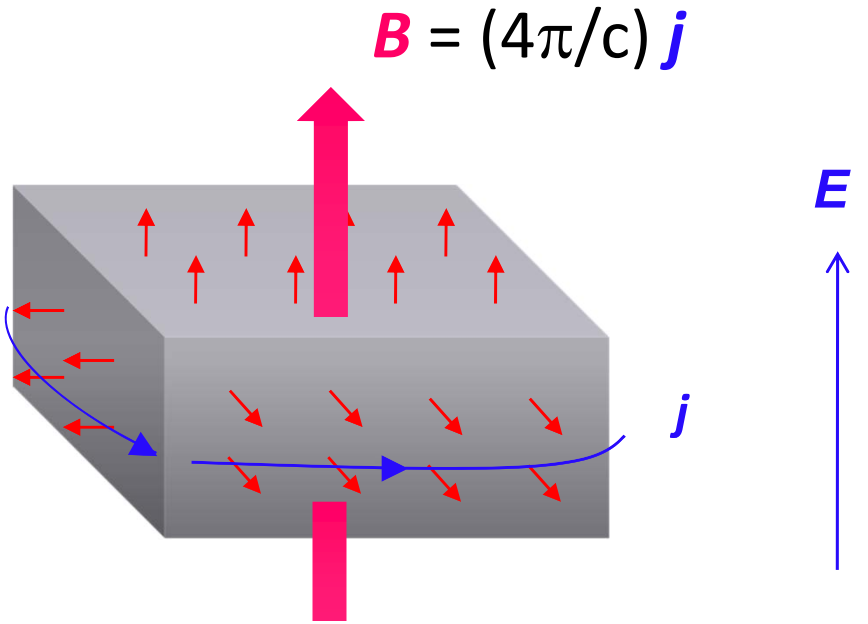
(magnetic moment) $M = \alpha_m E$ (electric field)

(electric polarization) $P = \alpha_e B$ (magnetic field)



Response to ElectroMagnetic fields

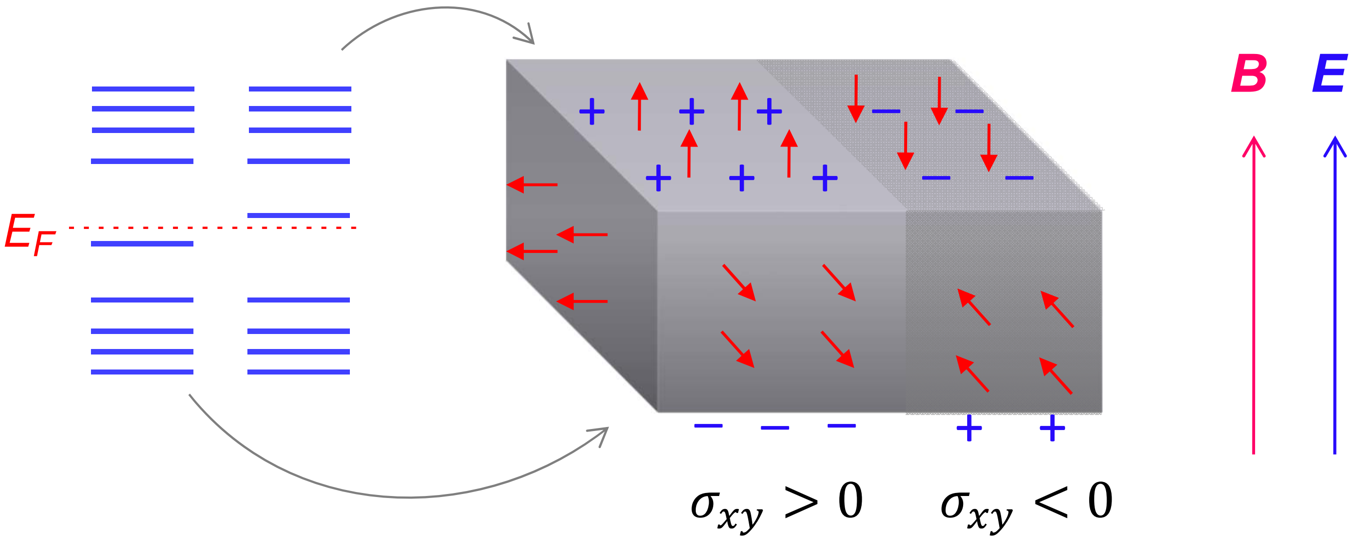
3D TI + magnetic impurities



$$\boldsymbol{M} = \frac{e^2}{2hc}\boldsymbol{E}$$

Qi, Hughes, Zhang ’08
Essin, Moore, Vanderbilt ’09

Response to ElectroMagnetic fields



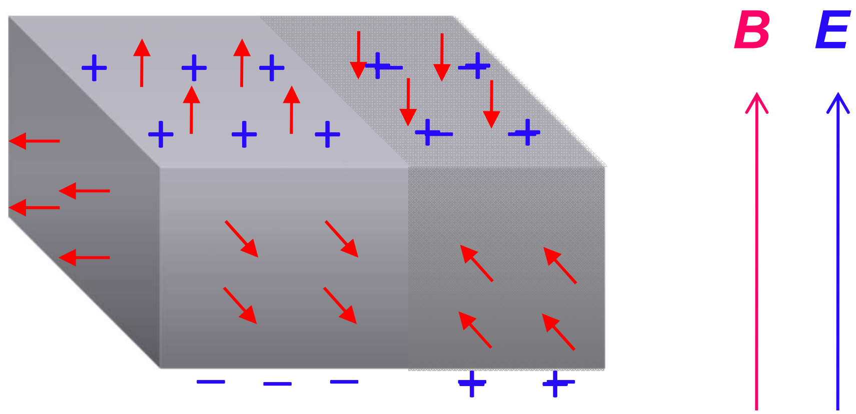
Surface QH states

$$j^{\mu} = \sigma_{xy}\epsilon^{\mu\nu\lambda}F_{\nu\lambda}$$

$$\begin{cases} \rho = \sigma_{xy} B_z \\ \boldsymbol{j} = \sigma_{xy}\boldsymbol{E} \times \hat{\boldsymbol{z}} \end{cases}$$

Response to ElectroMagnetic fields

$$E_{ME} = -\int d^3x \left(\frac{e^2}{4\pi\hbar}\right)\frac{\theta}{\pi}\boldsymbol{E}\cdot\boldsymbol{B}$$



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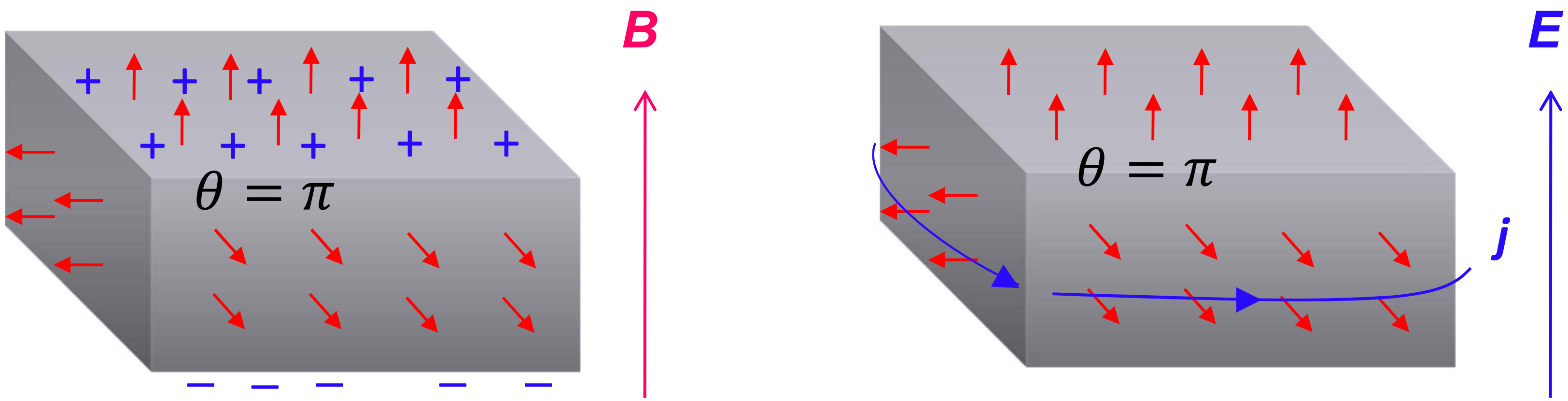


$$\mathbf{P} = -\frac{\delta E_{ME}}{\delta \mathbf{E}} = \frac{e^2}{4\pi\hbar}\frac{\theta}{\pi}\mathbf{B}$$

$$\mathbf{M} = -\frac{\delta E_{ME}}{\delta \mathbf{B}} = \frac{e^2}{4\pi\hbar}\frac{\theta}{\pi}\mathbf{E}$$

Qi, Hughes, Zhang '08

The Action Principle

$$S_{\text{Maxwell}} = -\int dt d^3 x \left(j^\mu A_\mu + \frac{1}{8\pi}(\boldsymbol{B}^2 - \boldsymbol{E}^2) \right)$$

↓

$$\frac{\delta S}{\delta A_\mu} = 0$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = 4\pi \boldsymbol{j}$$

The Action Principle

$$S_{\text{Maxwell}} = -\int dtd^3x\left(j^\mu A_\mu + \frac{1}{8\pi}(\boldsymbol{B}^2 - \boldsymbol{E}^2)\right)$$

$$S_\theta = \int dtd^3x\left(\frac{e^2}{2\pi h}\right)\theta\boldsymbol{E}\cdot\boldsymbol{B}$$

$$\frac{\delta S}{\delta A_\mu} = 0$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 4\pi\rho$$

$$\boldsymbol{\nabla}\times\boldsymbol{B} - \frac{\partial\boldsymbol{E}}{\partial t} = 4\pi\boldsymbol{j}$$

The Action Principle

$$S_{\text{Maxwell}} = -\int dt d^3 x \left(j^{\mu} A_{\mu} + \frac{1}{8\pi} (\boldsymbol{B}^2 - \boldsymbol{E}^2) \right)$$

$$S_{\theta} = \int dt d^3 x \left(\frac{e^2}{2\pi h} \right) \theta \boldsymbol{E} \cdot \boldsymbol{B}$$

↓

$$\frac{\delta S}{\delta A_{\mu}} = 0$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho \quad + 0$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = 4\pi \boldsymbol{j} \quad + 0$$

for constant θ

Axion term (θ term)

$$S_{\text{Maxwell}} = -\int dt d^3 x \left(j^\mu A_\mu + \frac{1}{8\pi}(\boldsymbol{B}^2 - \boldsymbol{E}^2) \right)$$

$$S_\theta = \int dt d^3 x \left(\frac{e^2}{2\pi h} \right) \theta \boldsymbol{E} \cdot \boldsymbol{B}$$

Peccei, Quinn 1977
Wilczek 1987

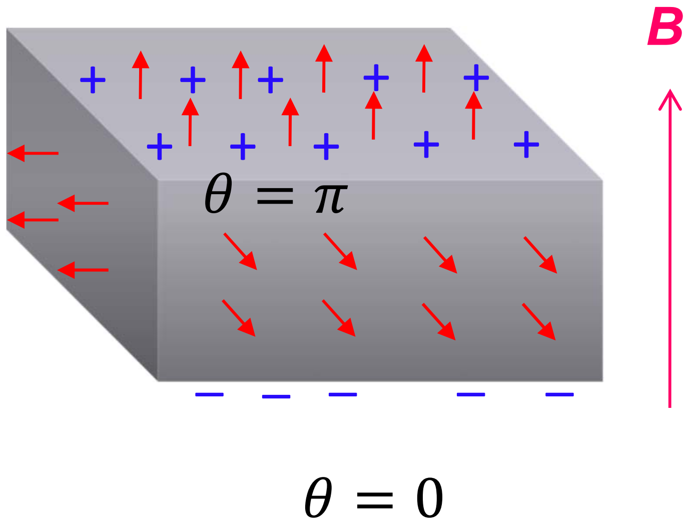
$$\theta = \theta(\boldsymbol{x}, t)$$

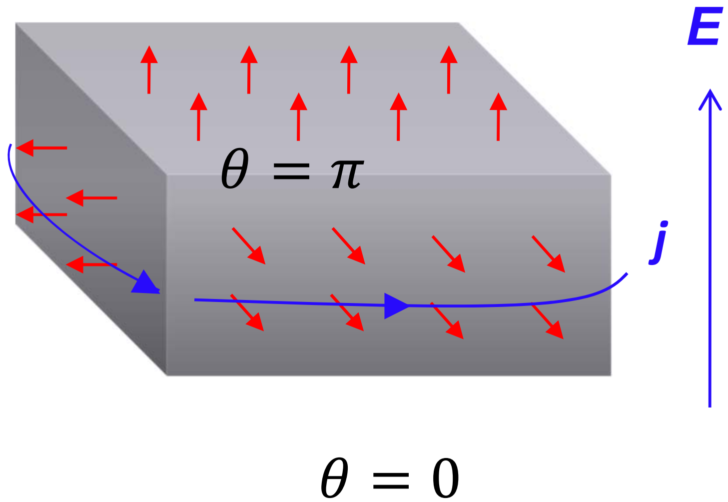
↓ $\frac{\delta S}{\delta A_\mu} = 0$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi \left[\rho + \frac{e^2}{2h} \boldsymbol{\nabla} \left(\frac{\theta}{\pi} \right) \cdot \boldsymbol{B} \right]$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = 4\pi \left[\boldsymbol{j} + \frac{e^2}{2h} \boldsymbol{\nabla} \left(\frac{\theta}{\pi} \right) \times \boldsymbol{E} + \frac{e^2}{2h} \left(\frac{\dot{\theta}}{\pi} \right) \boldsymbol{B} \right]$$

Axion term (θ term)





Qi, Hughes, Zhang 2008

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi \left[\rho + \frac{e^2}{2h} \boldsymbol{\nabla} \left(\frac{\theta}{\pi} \right) \cdot \boldsymbol{B} \right]$$

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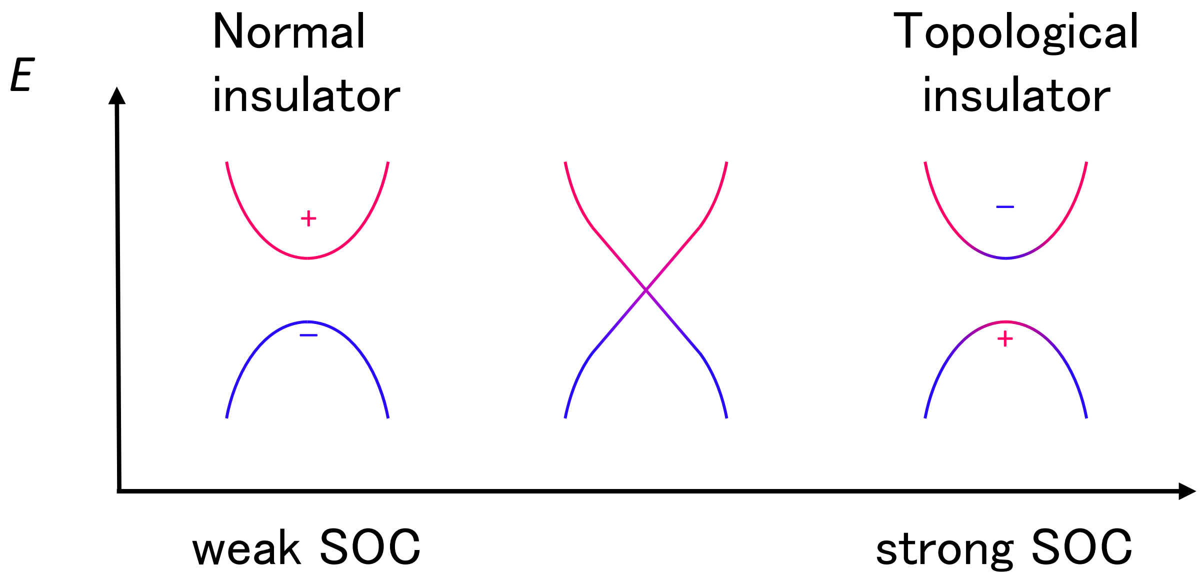
Summary

A topological insulator is a material with a finite bulk gap and gapless excitations at the surface.



It realizes novel magnetoelectric responese.

$$E_{ME} = -\int d^3x \left(\frac{e^2}{4\pi\hbar}\right)\frac{\theta}{\pi}\boldsymbol{E}\cdot\boldsymbol{B}$$